Coloring maps and graphs

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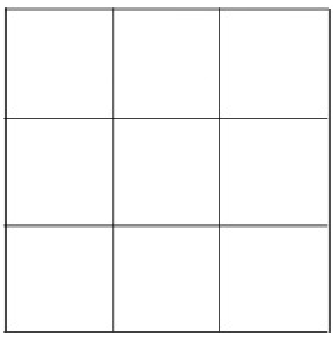
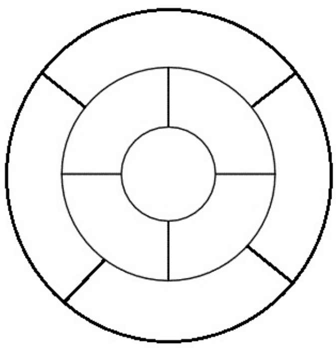
This document contains the outline of this activity. We will color a multitude of objects. The kids will receive a hand-out containing some of the more complicated coloring assignments. Slides will be used to state the problems.

The four-color theorem. (15 minutes) The four-color theorem states that any plane map can be colored using at most four colors in such a way that regions sharing a common boundary do not share the same color. In this unit, we will color a number of plane regions, and summarize our findings in the end, in the form of this theorem.

Question 1. *As a warm-up, color each object so that neighboring regions have different colors. Make sure to use the least amount of colors in each case.*





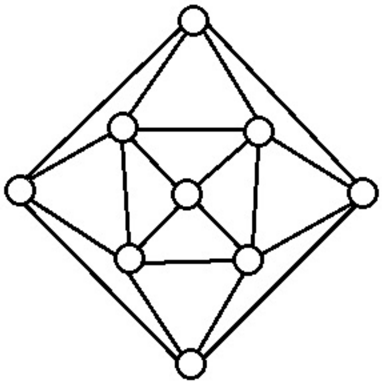
Question 2. *The following two maps illustrate part of the USA and Europe. Color both of them, using the least amount of colors, so that each neighboring region has different color. How many colors do you need?*

For all of the above maps, we only needed at most four colors. This empirical evidence suggests the following theorem:

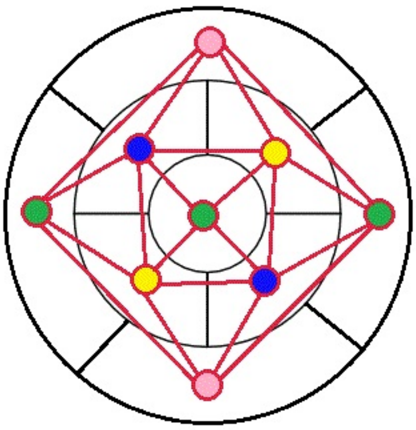
Theorem 3 (Appel, Haken, 1976). *Every map can be colored using at most four colors.*

Chromatic number of graphs. [15 minutes] A graph is a collection of nodes, with edges connecting them. A coloring of a graph is a choice of color for each node, so that no edge has the same color on each end.

Question 4. *How many colors do you need to color the following graph:*

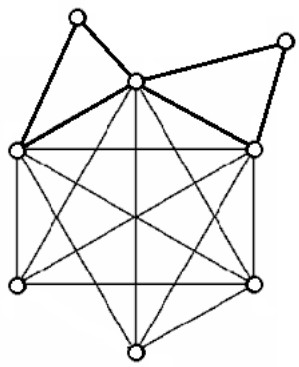


Now we realize that we already completed this task, as part of Question 1:



When it comes to coloring graphs, you may need more then four colors. See the following question:

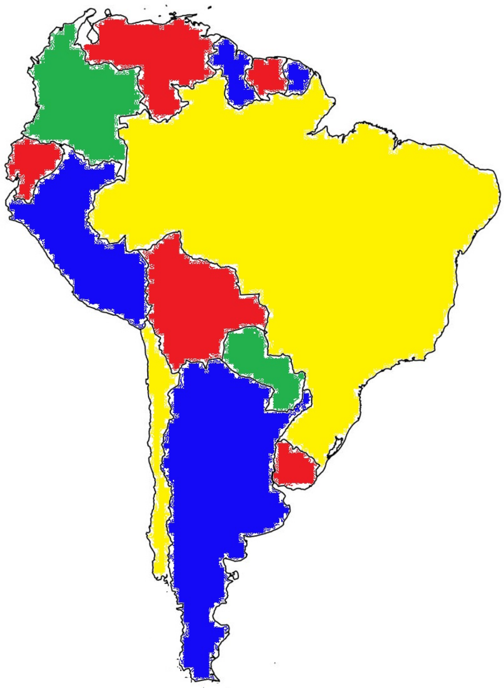
Question 5. *How many colors do you need to color the graph below?*



Variations on the theme of coloring. [20 minutes]

Question 6. *You are provided four colors, to color the map of South America. Every red region is worth 400\$. Every Yellow region is worth 300\$. Every blue region is worth 200\$. Every green region is worth 100\$.*

For example, the coloring below is worth $4 \times 400 + 2 \times 300 + 4 \times 200 + 2 \times 100 = 3200\$$. *How much money can we make on of this coloring operation?*



If we finish ahead of time here, we can do the same problem for the map of USA and Western Europe, from a previous problem, or we can simply try to minimize the cost.

Question 7. *Authorities have apprehended a number of high profile criminals and they want to put them into prison cells. Unfortunately, many of these criminals cannot stand each other and authorities risk violence if they put them in the same cell:*

Billy the Kid can not stand AC, BP, JJ, LL
Jesse James can not stand BP, JH, BK, LL,MC.
Jimmy Hoffa can not stand JJ, BP, AC, LL.
Lucky Luciano can not stand BK, AC, JH, JJ, MC.
Mickey Cohen can not stand JJ, LL.
Al Capone can not stand WB, BP, JH, LL, BK.
Wild Butch can not stand AC,BP.
Bonnie Parker can not stand WB, AC, JH, JJ, BK.

How many cells do we need for these criminals, while making sure that peace is preserved? What does this have to with graph coloring?

Here is a possible solution (3 prison cells):