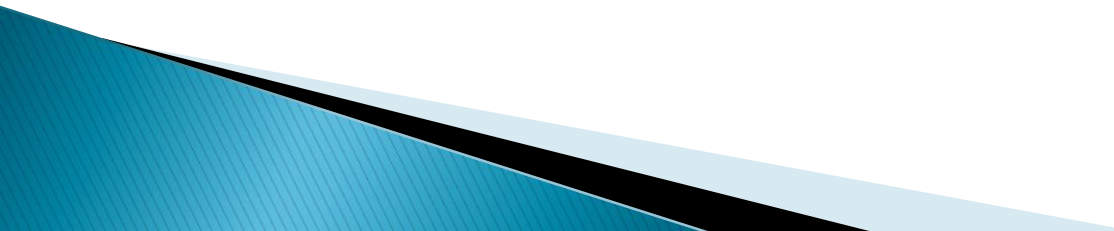


Probability Paradoxes

The DC Math Circle

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Monty Hall paradox

- ▶ It's "Let's Make a Deal". The grand prize is behind one of three curtains. You choose a curtain. Monty Hall opens another of the curtains to reveal a booby prize. He then offers you the option of switching to the remaining curtain. Should you switch or stay put?
- 

Answer

- ▶ Switch! You will get the grand prize with probability $2/3$ if you switch, $1/3$ if you stay.

Warm-up

- ▶ If you select at random a person with exactly two children at least one of whom is a son, what is the probability that the person has two sons?

Answer

- ▶ 1/3.
- ▶ Why?

		Second Child	
		B	G
First Child	B	BB	BG
	G	GB	GG

Variation

- ▶ If you select at random a person with exactly two children at least one of whom is a son **named Poindexter**, what is the probability that the person has two sons?

Answer

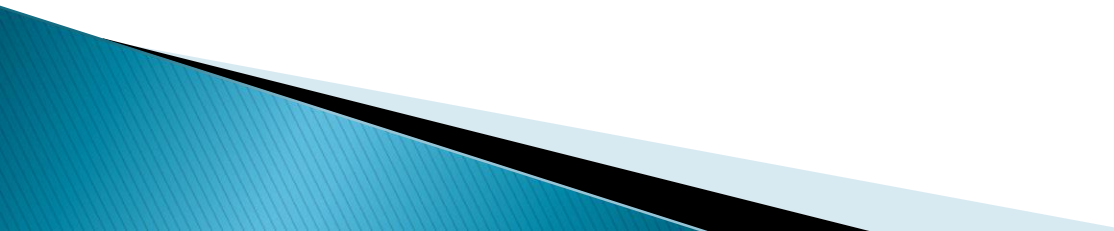
- ▶ Nearly 1 / 2.
- ▶ Why?

		Second Child		
		P	B	G
First Child	P	PP	PB	PG
	B	BP	BB	BG
	G	GP	GB	GG

Two envelopes paradox

- ▶ I put x dollars in one envelope and $2x$ dollars in another. You don't know x . I hand you one envelope and you open it. Now I invite you to keep the contents or switch to the other envelope. Should you stay or switch?

Answer?

- ▶ Switch, of course. You have a 50% chance of doubling your money and a 50% chance of losing half your money. Good deal!
 - ▶ It doesn't matter. The two envelopes are the same.
 - ▶ What gives?
- 

St. Petersburg Paradox

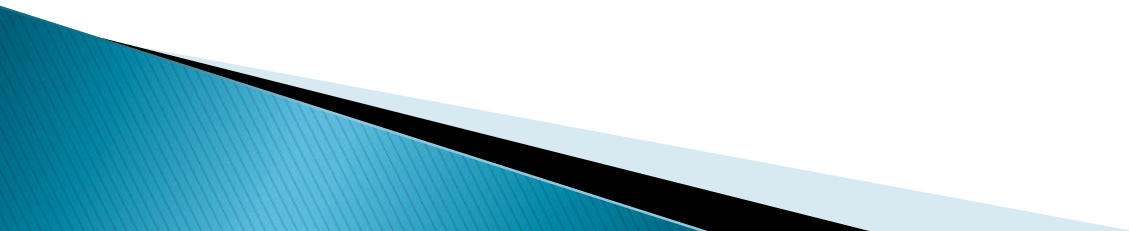
- ▶ Flip a coin until you get heads. If it took n tries, I'll pay you 2^n dollars.
- ▶ But, hold on: There is an entrance fee. Do you still want to play?

Solution?

- ▶ Yes, you should play, no matter how large the entrance fee! Your expected payoff is

$$(1/2) \cdot 2 + (1/4) \cdot 4 + (1/8) \cdot 8 + (1/16) \cdot 16 + \dots$$

- ▶ Huh?



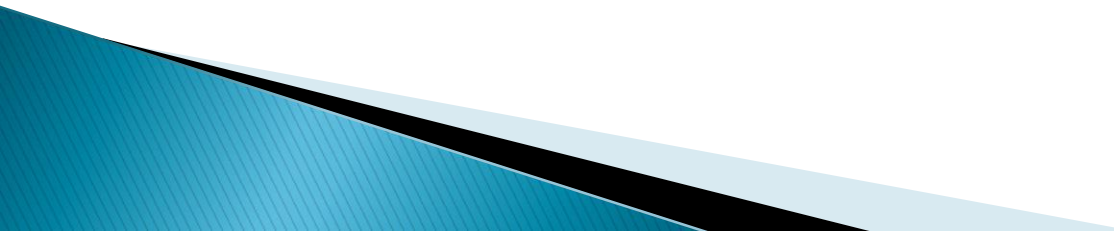
The next card is red

- ▶ I deal out the cards of a regular deck. At any moment, you may guess that the next card will be red. What is your best strategy? How much better than 50:50 can you do?

Answer

- ▶ You can't do better than 50–50.
- ▶ How to see this? Suppose that when you guess that the next card is red, I deal from the *bottom* of the deck instead of the top. Would that change anything?

A tempting investment

- ▶ An investment offers a 50% chance of increasing by 60% and a 50% chance of decreasing by 40%. Would you take that gamble?
 - ▶ Would you take it repeatedly?
- 

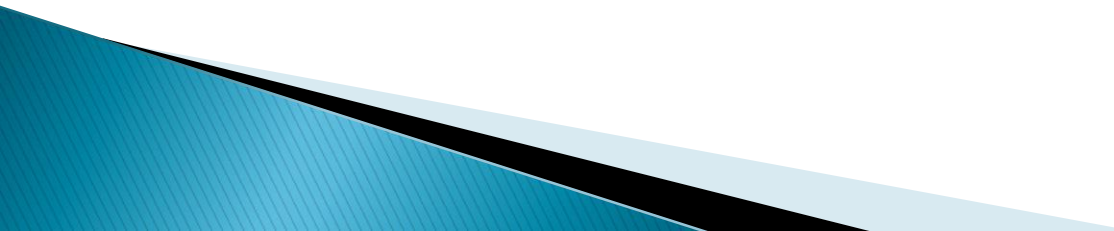
Answer?

- ▶ Yes, because each gamble has an expected gain (of 10%).
- ▶ No, because you'll win about half the time, but the gains don't compensate for the losses, because

$$(0.6)(1.6) = 0.96 < 1.$$

- ▶ So, which is it? Yes or no?

Simpson's Paradox


- ▶ Player A has a higher batting average this year than Player B. Player A had a high batting average last year than Player B. Must Player A have a higher batting average over the two years than Player B?
- 

Answer

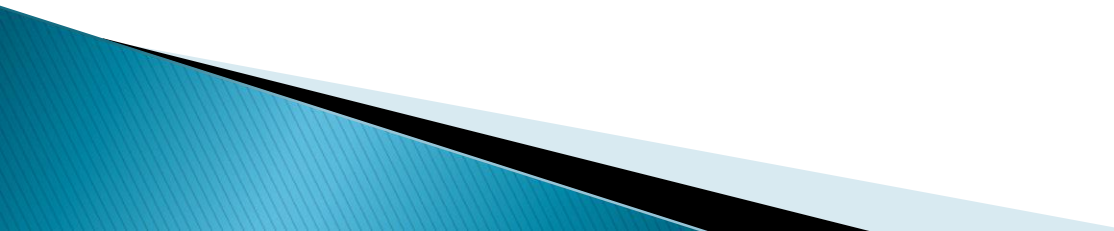
	1995	1996	Combined
Derek Jeter	.250	.314	.310
David Justice	.253	.321	.270

	1995	1996	Combined
Derek Jeter	$12/48 = .250$	$183/582 = .314$	$195/630 = .310$
David Justice	$104/411 = .253$	$45/140 = .321$	$149/551 = .270$

The Pop Quiz

- ▶ Your teacher say: “Next week, we will have a quiz. I will not announce on which day the quiz will be, nor will you be able to deduce on which day the quiz will be.”
 - ▶ You reason: “Not Friday.” “Therefore not Thursday.” “Therefore not Wednesday.” “Therefore not Tuesday.” “Therefore not Monday!”
- 

What actually happens:

- ▶ The quiz is given on Tuesday, say.
 - ▶ You didn't expect it.
 - ▶ The teacher was right!
- 

Nontransitive dice

- ▶ Suppose we have three dice:
 - Die A has 2,2,4,4,9,9
 - Die B has 1,1,6,6,8,8, and
 - Die C has 3,3,5,5,7,7.
- ▶ We each choose a die and whoever rolls higher wins. Which die is best?

Some trivia questions:

- ▶ What is the only state other than Alaska that has never recorded a temperature above 100 degrees?
 - ▶ The two most distant points in the 50 US states are in what two states?
 - ▶ What is the largest US city that is not the largest US city with its name?
- 