Shapes and Symmetry

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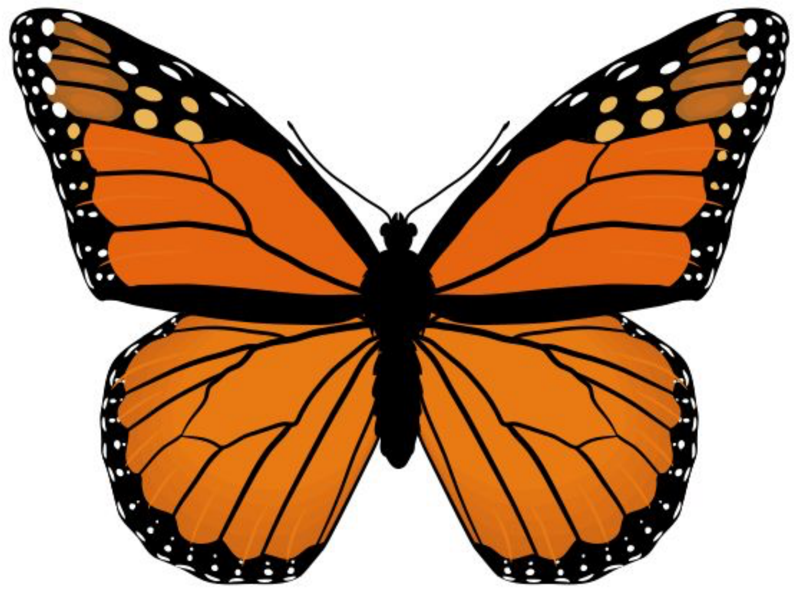
Symmetry



- I study symmetry of different kinds of shapes.
- Normally, if you do some transformation to a shape (like rotating it), you can tell because the shape looks different.
- But if the shape looks the same even after the transformation, the shape has *symmetry*—it's *symmetric*.

Mattress Flipping

A mattress needs to be flipped from time to time, or else it gets lumpy and uncomfortable.

You can:

- “flip” it side-to-side (then it goes from position A to position C),
- “flop” it head-to-toe,
- “flipflop” it by flipping, then flopping,
- “flopflip” it by flopping, then flipping (same as a flipflop),
- be lazy and do nothing!

Mattress Flipping

You can also do a combination of things to the mattress, like flipping, then flopflipping. If you do, then the new position of the mattress could have been achieved by doing just one transformation! Make a table that shows what happens to the mattress after I move it in *two* of these ways.

then / first	**Nothing**	**Flip**	**Flop**	**Flopflip**
Nothing				
Flip				
Flop				
Flopflip				

Mattress Flipping

You can also do a combination of things to the mattress, like flipping, then flopflipping. If you do, then the new position of the mattress could have been achieved by doing just one transformation! Make a table that shows what happens to the mattress after I move it in *two* of these ways.

then / first	**Nothing**	**Flip**	**Flop**	**Flopflip**
Nothing	Nothing	Flip	Flop	Flopflip
Flip	Flip	Nothing	Flopflip	Flop
Flop	Flop	Flopflip	Nothing	Flip
Flopflip	Flopflip	Flop	Flip	Nothing

Mattress Flipping

Note that a flopflip is the same as a flat 180° rotation of the mattress!

What if the mattress is a square and not a rectangle?

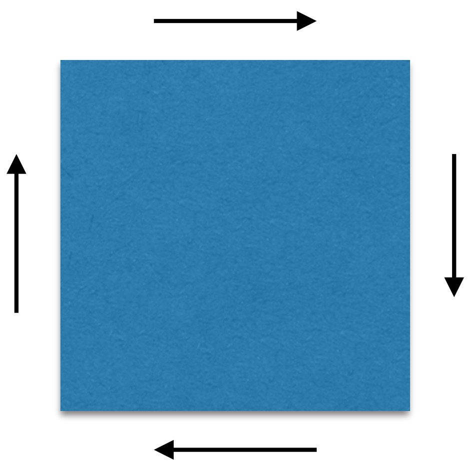
Then in addition to the previous rotations, you can also perform 90° rotations.

How many symmetries in total does the mattress have now?

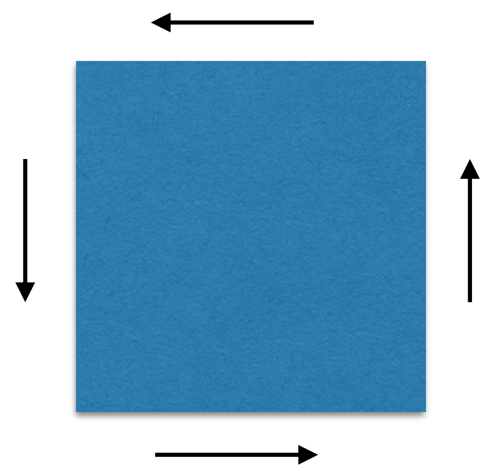
Mattress Flipping

So we have these two new rotations:

Turn: 90° clockwise rotation



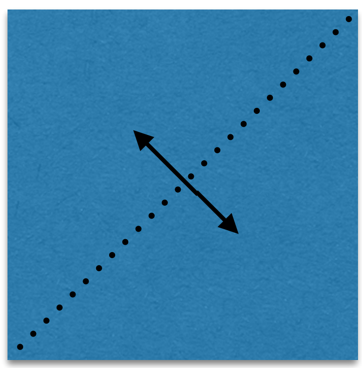
Nurt: 90° counterclockwise rotation



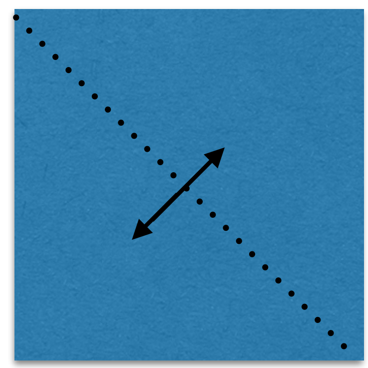
Mattress Flipping

And we have two more reflections across diagonals:

Criss: flipping across a diagonal



Cross: flipping across the other diagonal



Mattress Flipping

Complete this expanded table that shows what happens to the mattress after you do a combination of two of these moves.

then / first	Nothing	Flip	Flop	Flopflip	Turn 90°	Nurt -90°	Criss	Cross
Nothing	Nothing	Flip	Flop	Flopflip	Turn	Nurt	Criss	Cross
Flip	Flip	Nothing	Flopflip	Flop				
Flop	Flop	Flopflip	Nothing	Flip				
Flopflip	Flopflip	Flop	Flip	Nothing				
Turn 90°	Turn							
Nurt -90°	Nurt							
Criss	Criss							
Cross	Cross							

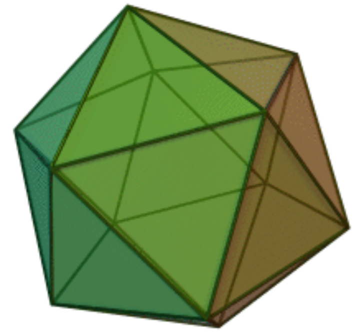
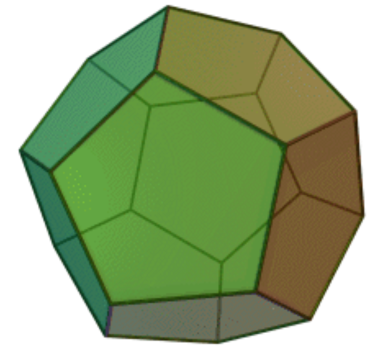
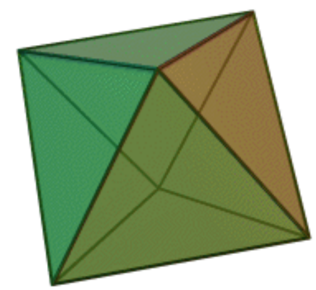
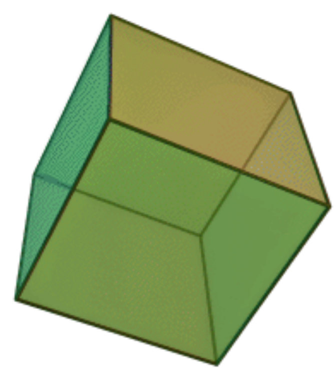
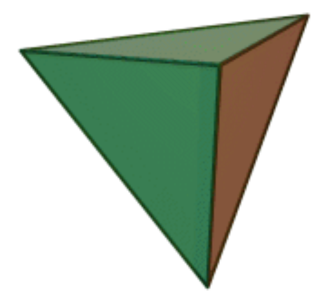
Mattress Flipping

Complete this expanded table that shows what happens to the mattress after you do a combination of two of these moves.

then / first	Nothing	Flip	Flop	Flopflip	Turn 90°	Nurt -90°	Criss	Cross
Nothing	Nothing	Flip	Flop	Flopflip	Turn	Nurt	Criss	Cross
Flip	Flip	Nothing	Flopflip	Flop	Criss	Cross	Turn	Nurt
Flop	Flop	Flopflip	Nothing	Flip	Cross	Criss	Nurt	Turn
Flopflip	Flopflip	Flop	Flip	Nothing	Nurt	Turn	Cross	Criss
Turn 90°	Turn	Cross	Criss	Nurt	Flopflip	Nothing	Flop	Flip
Nurt -90°	Nurt	Criss	Cross	Turn	Nothing	Flopflip	Flip	Flop
Criss	Criss	Nurt	Turn	Cross	Flop	Flip	Nothing	Flopflip
Cross	Cross	Turn	Nurt	Criss	Flip	Flop	Flopflip	Nothing

Regular Polyhedra

- A polyhedron is a three-dimensional shape with flat and straight edges.
- A polyhedron is regular if all of its faces are identical regular polygons—all the edges have the same length make the same angles.
- There are only 5 regular polyhedra: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron.

Regular Polyhedra

Complete the following table showing how many faces, edges and vertices the regular polyhedra have.

Polyhedron	Faces	Edges per Face	Edges	Edges Meeting at any Vertex	Vertices
Tetrahedron	4	3		3	
Cube	6	4		3	
Octahedron	8	3		4	
Dodecahedron	12	5		3	
Icosahedron	20	3		5	

Regular Polyhedra

Complete the following table showing how many faces, edges and vertices the regular polyhedra have.

Polyhedron	Faces	Edges per Face	Edges	Edges Meeting at any Vertex	Vertices
Tetrahedron	4	3	$(4\times3)\div2 = 6$	3	$(6\times2)\div3 = 4$
Cube	6	4	$(6\times4)\div2 = 12$	3	$(12\times2)\div3 = 8$
Octahedron	8	3	$(8\times3)\div2 = 12$	4	$(12\times2)\div4 = 6$
Dodecahedron	12	5	$(12\times5)\div2 = 30$	3	$(30\times2)\div3 = 20$
Icosahedron	20	3	$(20\times3)\div2 = 30$	5	$(30\times2)\div5 = 12$

Symmetries of Regular Polyhedra

How many different symmetries do each of the regular polyhedra have? Complete the following table.

Polyhedron	Faces	Edges per Face	Edges	Edges Meeting at any Vertex	Vertices	Symmetries
Tetrahedron	4	3	6	3	4	
Cube	6	4	12	3	8	
Octahedron	8	3	12	4	6	
Dodecahedron	12	5	30	3	20	
Icosahedron	20	3	30	5	12	

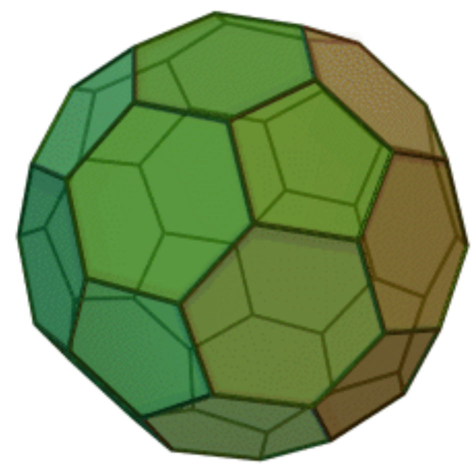
Symmetries of Regular Polyhedra

How many different symmetries do each of the regular polyhedra have? Complete the following table.

Polyhedron	Faces	Edges per Face	Edges	Edges Meeting at any Vertex	Vertices	Symmetries
Tetrahedron	4	3	6	3	4	$4 \times 3 = 12$
Cube	6	4	12	3	8	$6 \times 4 = 24$
Octahedron	8	3	12	4	6	$8 \times 3 = 24$
Dodecahedron	12	5	30	3	20	$12 \times 5 = 60$
Icosahedron	20	3	30	5	12	$20 \times 3 = 60$

Soccer Ball

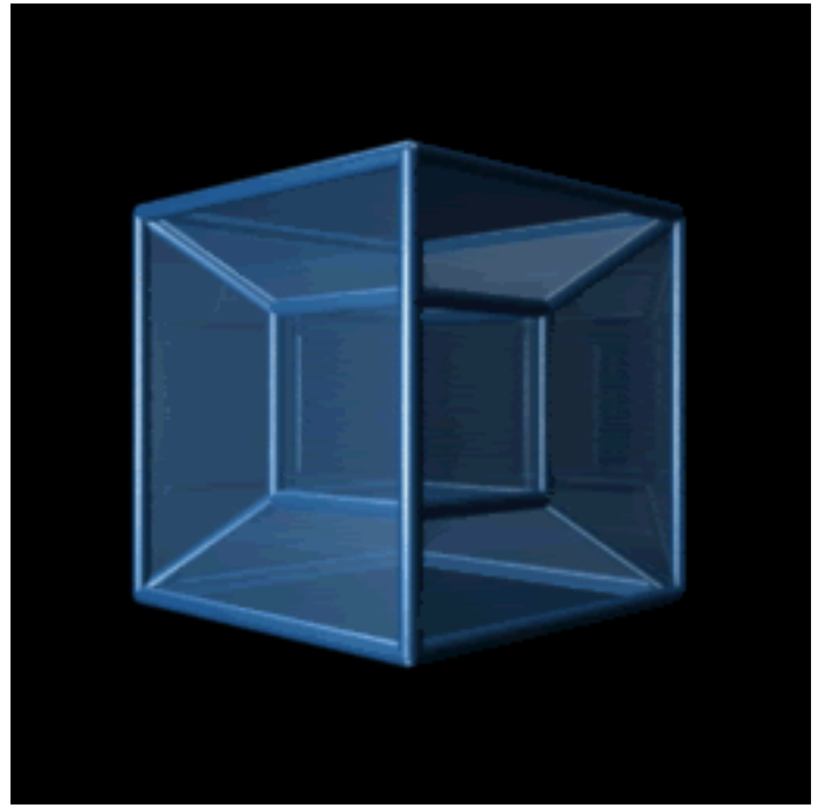
- Otherwise known as a truncated icosahedron.
- Some faces are pentagons, some are hexagons, but they occur in a nice pattern.
- Each pentagon has 5 hexagon neighbors.
- Each hexagon has 3 pentagon neighbors and 3 hexagon neighbors, in alternating order.



Soccer Ball

- If there are 12 pentagonal faces, how many symmetries does the soccer ball have?
- How many hexagonal faces are there?

Hypercubes and Symmetry



A hypercube is the four-dimensional version of a cube!

You can make one by:

- Taking two different cubes, and placing one above the other (in the fourth dimension!).
- Making new edges from each vertex in the top cube to the corresponding one in the bottom cube.
- An edge below together with the corresponding edge above and the two new edges on either side make a new square face for the hypercube.
- A face below together with the corresponding face above and the four new faces on all sides makes a cubic hyperface for the hypercube.

Hypercubes and Symmetry

Complete this table showing the number of vertices, edges, faces, and hyperfaces on a hypercube. How many symmetries does the hypercube have?

Shape	Vertices	Edges	Faces (Squares)	Hyperfaces (Cubes)
Cube	8	12	6	1
Hypercube				

Hypercubes and Symmetry

Complete this table showing the number of vertices, edges, faces, and hyperfaces on a hypercube. How many symmetries does the hypercube have?

Shape	Vertices	Edges	Faces (Squares)	Hyperfaces (Cubes)
Cube	8	12	6	1
Hypercube	8+8 = 16	12+12+8 = 32	6+6+12 = 24	1+1+6 = 8