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Multivariate Stable Distributions

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Chapter 1
Basic Properties of Multivariate Stable Distributions

This book is on multivariate stable distributions; it builds on a previous volume on univariate stable laws. As in one dimension, these distributions have heavy tails and may be skewed. The surprise here is that the possible dependence structures are much richer than the elliptically contoured Gaussian case. This is interesting because it allows a much wider range of dependence structures. Unfortunately, it also requires a more complicated theory, especially in the non-symmetric case.

As in Volume 1, this introductory chapter discusses the basic facts about multivariate stable distributions without proofs. The next chapter contains proofs and more technical details about these distributions. Since the dependence structure is generally quite complex, it is discussed in Chapter 3, which also includes definitions of metrics for closeness of multivariate stable distributions. Chapter 4 discusses the statistical estimation of multivariate stable laws, and diagnostics for assessing the fit.

1.1 Definition of jointly stable

Vectors in d-dimensional space will generally be columns:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} = (x_1, x_2, \ldots, x_d)^T \in \mathbb{R}^d.$$

Comparisons between vectors are defined coordinate-wise, e.g. $\mathbf{x} \leq \mathbf{y}$ means $x_j \leq y_j$ for all $j = 1, \ldots, d$.

Definition 1.1 A nondegenerate d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)^T$ is *stable* or *stable in the broad sense* if for $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ independent copies of $\mathbf{X}$ and any positive constants a and b,

$$a\mathbf{X}^{(1)} + b\mathbf{X}^{(2)} \stackrel{d}{=} c\mathbf{X} + \mathbf{d}, \tag{1.1}$$

for some positive $c = c(a, b)$ and some $\mathbf{d} = \mathbf{d}(a, b) \in \mathbb{R}^d$. The random vector is *strictly stable* or *stable in the narrow sense* if (1.1) holds with $\mathbf{d} = 0$ for all choices of a and b. A random vector is *symmetric stable* if it is stable and symmetrically distributed around 0, e.g. $\mathbf{X} \stackrel{d}{=} -\mathbf{X}$.

Note that symmetry for a multivariate stable distribution means more than just the components separately being symmetric; when the components are dependent, the margins can be symmetric even though the joint distribution is non-symmetric, see Problem 1.1. Also, multivariate symmetric stable includes much more than radially symmetric (isotropic): there are many non-isotropic symmetric stable distributions.

As in the one dimensional case, there are equivalent definitions of stability for random vectors. Here is the multivariate version of Definition I. 1.2.

Definition 1.2 A nondegenerate d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)^T$ is *stable* or *stable in the broad sense* if for all $n > 1$, there exists constants $c_n > 0$ and a vector $\mathbf{d}_n \in \mathbb{R}^d$ such that

$$\mathbf{X}^{(1)} + \cdots + \mathbf{X}^{(n)} \stackrel{d}{=} c_n\mathbf{X} + \mathbf{d}_n, \tag{1.2}$$

where $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(n)}$ are independent copies of $\mathbf{X}$. The random vector is *strictly stable* or *stable in the narrow sense* if (1.2) holds with $\mathbf{d}_n = 0$ for all $n > 1$.

We will show that many of the properties of univariate stable laws hold in the multivariate case, e.g. sums of α-stable vectors are α-stable. Later we will show that sums of dependent univariate α-stable variables are stable, after we know how to characterize dependence. But first, some examples of stable random vectors are given.

Example 1.1 Let $X_1, \ldots, X_n$ be n independent univariate α-stable random variables and let $\mathbf{t}_1, \ldots, \mathbf{t}_n$ be arbitrary points in $\mathbb{R}^d$. Define

$$\mathbf{X} = \sum_{j=1}^{n} X_j \mathbf{t}_j. \tag{1.3}$$

We claim that $\mathbf{X}$ is stable. Let $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ be independent copies of $\mathbf{X}$. This is equivalent to $\mathbf{X}^{(1)} \stackrel{d}{=} \sum_{j=1}^{n} X_j^{(1)} \mathbf{t}_j$ and $\mathbf{X}^{(2)} \stackrel{d}{=} \sum_{j=1}^{n} X_j^{(2)} \mathbf{t}_j$. Then for any $a, b > 0$,

$$\begin{aligned} a\mathbf{X}^{(1)} + b\mathbf{X}^{(2)} &\stackrel{d}{=} a \sum_{j=1}^{n} X_j^{(1)} \mathbf{t}_j + b \sum_{j=1}^{n} X_j^{(2)} \mathbf{t}_j = \sum_{j=1}^{n} \left(aX_j^{(1)} + bX_j^{(2)} \right) \mathbf{t}_j \\ &= \sum_{j=1}^{n} \left(c(a, b) X_j + d_j \right) \mathbf{t}_j = c(a, b) \sum_{j=1}^{n} X_j \mathbf{t}_j + \sum_{j=1}^{n} d_j \mathbf{t}_j \\ &= c(a, b)\mathbf{X} + \mathbf{d}, \end{aligned}$$

where the middle step holds by the stability property of the univariate X_j's. Note that $c(a,b) = (a^\alpha + b^\alpha)^{1/\alpha}$ is the same for all j.

It will be shown in Section 3.6 that every stable random vector is the limit in distribution of stable random vectors of form (1.3) plus a shift.

Example 1.2 An important special case of Example 1.1 is when $n = d$ and $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_d$ are the standard basis vectors $\mathbf{e}_1 = (1,0,0,\ldots,0)^T, \mathbf{e}_2 = (0,1,0,\ldots,0)^T$, $\ldots, \mathbf{e}_d = (0,0,\ldots\ldots,1)^T$. Then

$$\mathbf{X} = (X_1, X_2, \ldots, X_d)^T = \sum_{j=1}^{d} X_j \mathbf{e}_j$$

is stable with independent components.

Example 1.3 Gaussian and sub-Gaussian/elliptically contoured random vectors. Let $\mathbf{G} \sim N(0,\Sigma)$ be any multivariate normal with mean 0. First, it is straightforward that $\mathbf{G}$ is stable:

$$a\mathbf{G}^{(1)} + b\mathbf{G}^{(2)} \stackrel{d}{=} (a^2 + b^2)^{1/2}\mathbf{G}, \tag{1.4}$$

where $\mathbf{G}^{(1)}$ and $\mathbf{G}^{(2)}$ are independent copies of $\mathbf{G}$. Hence any Gaussian random vector is stable.

Now fix $0 < \alpha < 2$ and let univariate $P \sim \mathbf{S}(\alpha/2, 1; 1)$ be a univariate positive stable random variable that is independent of G; it will be shown that $\mathbf{X} = P^{1/2}\mathbf{G}$ is α-stable. Let $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ be independent copies of $\mathbf{X}$, then $\mathbf{X}^{(1)} \stackrel{d}{=} P_1^{1/2}\mathbf{G}^{(1)}$ and $\mathbf{X}^{(2)} \stackrel{d}{=} P_2^{1/2}\mathbf{G}^{(2)}$ where $\mathbf{G}^{(1)}$ and $\mathbf{G}^{(2)}$ are independent copies of $\mathbf{G}$. Using the independence of $P_1, P_2, \mathbf{G}^{(1)}$, and $\mathbf{G}^{(2)}$, (1.4) and properties of the one dimensional stable P_1, P_2, P, shows that for $a, b > 0$,

$$\begin{aligned} a\mathbf{X}^{(1)} + b\mathbf{X}^{(2)} &\stackrel{d}{=} aP_1^{\frac{1}{2}}\mathbf{G}^{(1)} + bP_2^{\frac{1}{2}}\mathbf{G}^{(2)} \stackrel{d}{=} (a^2P_1 + b^2P_2)^{1/2}\mathbf{G} \\ &\stackrel{d}{=} \left((a^{2(\alpha/2)} + b^{2(\alpha/2)})^{2/\alpha}P\right)^{1/2}\mathbf{G} = (a^\alpha + b^\alpha)^{1/\alpha}P^{1/2}\mathbf{G} \\ &\stackrel{d}{=} (a^\alpha + b^\alpha)^{1/\alpha}\mathbf{X}. \end{aligned}$$

A stable vector $\mathbf{X} = P^{1/2}\mathbf{G}$ is called *sub-Gaussian*, with P called the *stable subordinator* and $\mathbf{G}$ called the *dominating* or *governing* random vector. (Note: some authors use the term sub-Gaussian to mean something different, e.g. a random variable, vector or process that has exponential moments.)

In Section 1.3 of Nolan (2020b) the notation $X \sim \mathbf{S}(\alpha, \beta, \gamma, \delta; 1)$ is used to specify that X is a univariate stable distribution with index of stability α, skewness β, scale γ, and location δ in the 1-parameterization.

Example 1.4 Let $X_1 \sim \mathbf{S}(\alpha_1, \beta_1, \gamma_1, \delta_1; 1)$ and $X_2 \sim \mathbf{S}(\alpha_2, \beta_2, \gamma_2, \delta_2; 1)$ be independent with $\alpha_1 \neq \alpha_2$. Problem 1.2 shows that the vector $\mathbf{X} = (X_1, X_2)^T$ is not multivariate stable.

The phrase *jointly stable* is sometimes used to stress the fact that the definition forces all the components X_j to have the same index of stability. This follows from the following theorem which shows that all components are univariate stable and that they all have the same index of stability. There is a different concept, called *marginally stable* or *operator stable*, that allows the components of $\mathbf{X}$ to have different indices of stability. These arise when a, b, and c are allowed to be matrices in Definition 1.1, see Section 7.3.

We will use linear combinations of the coordinates of $\mathbf{X}$ regularly; they can be concisely described in vector notation for the inner or dot product: $\langle \mathbf{u}, \mathbf{X} \rangle = u_1 X_1 + u_2 X_2 + \cdots + u_d X_d$ is the projection of $\mathbf{X}$ in the direction $\mathbf{u}$.

Theorem 1.1 ***Projection Characterization of Joint Stability***
(a) Let $\mathbf{X}$ be a jointly stable random vector. Then for every $\mathbf{u} \in \mathbb{R}^d$, the one dimensional projection $\langle \mathbf{u}, \mathbf{X} \rangle = \sum u_i X_i$ is a univariate stable random variable with the same index α.
(b) Conversely, suppose $\mathbf{X}$ is a random vector with the property that every one-dimensional projection $\langle \mathbf{u}, \mathbf{X} \rangle$ is one dimensional stable, say $\langle \mathbf{u}, \mathbf{X} \rangle \sim \mathbf{S}(\alpha(\mathbf{u}), \beta(\mathbf{u}), \gamma(\mathbf{u}), \delta(\mathbf{u}); 1)$. Then there is one α that is the index of all projections, i.e. $\alpha(\mathbf{u}) = \alpha$ is constant. If $\alpha \geq 1$, then $\mathbf{X}$ is stable. If $\alpha < 1$ and the location parameter function $\delta(\mathbf{u})$ and the vector of location parameters $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_d)$ of all the components $X_1, X_2, \ldots, X_d$ are related by

$$\delta(\mathbf{u}) = \langle \mathbf{u}, \boldsymbol{\delta} \rangle, \tag{1.5}$$

then $\mathbf{X}$ is stable.
c) A stable random vector $\mathbf{X}$ is strictly stable if and only if all one dimensional projections $\langle \mathbf{u}, \mathbf{X} \rangle$ are strictly stable.

It will be shown below that (1.5) holds automatically when $\alpha > 1$ and that it is necessary when $\alpha \neq 1$, so it cannot be dropped. Example 2.1 shows there are random vectors where all one dimensional projections are $\alpha-$stable with $\alpha < 1$, but (1.5) fails and $\mathbf{X}$ is not jointly stable. Corollaries 1.1 and 1.2 gives condition for strict stability in terms of the spectral measure and the projection parameter functions.

Fix any $\mathbf{u} \in \mathbb{R}^d$. Then $Y = Y(\mathbf{u}) := \langle \mathbf{u}, \mathbf{X} \rangle$ is univariate stable by the preceding theorem. Taking the inner product of $\mathbf{u}$ with both sides of (1.1) and distributing shows that

$$aY^{(1)} + bY^{(2)} \stackrel{d}{=} cY + \langle \mathbf{u}, \mathbf{d} \rangle.$$

Since Y is univariate stable, the proof on page I.58 shows that the constant c must be of the form $c = (a^\alpha + b^\alpha)^{1/\alpha}$. This allows us to describe another class of stable random vectors.

Example 1.5 Sub-stable random vectors. A generalization of sub-Gaussian random vectors is to let $0 < \alpha_2 < \alpha_1 \leq 2$, $\mathbf{Y}$ be any strictly α_1-stable random vector, and R a univariate positive (α_2/α_1)-stable random variable. Then $\mathbf{X} := R^{1/\alpha_1}\mathbf{Y}$ is a strictly α_2-stable random vector: using the same notation and argument as in the sub-Gaussian case,

$$
\begin{aligned}
a\mathbf{X}^{(1)} + b\mathbf{X}^{(2)} &\stackrel{d}{=} aR_1^{1/\alpha_1}\mathbf{Y}^{(1)} + bR_2^{1/\alpha_1}\mathbf{Y}^{(2)} \stackrel{d}{=} ((aR_1^{1/\alpha_1})^{\alpha_1} + (bR_2^{1/\alpha_1})^{\alpha_1})^{1/\alpha_1}\mathbf{Y} \\
&= (a^{\alpha_1}R_1 + b^{\alpha_1}R_2)^{1/\alpha_1}\mathbf{Y} \\
&\stackrel{d}{=} \left(\left(a^{\alpha_1(\alpha_2/\alpha_1)} + b^{\alpha_1(\alpha_2/\alpha_1)}\right)^{\alpha_1/\alpha_2} R\right)^{1/\alpha_1}\mathbf{Y} \\
&= (a^{\alpha_2} + b^{\alpha_2})^{1/\alpha_2} R^{1/\alpha_1}\mathbf{Y} \stackrel{d}{=} (a^{\alpha_2} + b^{\alpha_2})^{1/\alpha_2}\mathbf{X}.
\end{aligned}
$$

Such a stable vector $\mathbf{X}$ is called *sub-stable*, with R called the *stable subordinator* and $\mathbf{Y}$ called the *dominating* or *governing* random vector.

1.2 Representations of jointly stable vectors

In this section we describe five ways to specify a multivariate stable law: using the projection parameter functions, using a spectral measure, using stochastic integrals, using a stochastic series representation, and using convex sets.

These different representations each have advantages and disadvantages. They all describe the joint behavior of the components, albeit in different ways. In the projection representation, the dependence is contained in the joint behavior of the projection parameter functions; in the spectral measure representation the dependence is described by the way mass is spread on the unit sphere; in the stochastic integral representation or the series representation the dependence is described by the joint behavior of d functions $f_1, \ldots, f_d$ and a control measure; in the last description, the shape of a particular convex set determines the distribution.

Unfortunately, none of the representations are very intuitive, e.g. understanding the density surface is not simple in any representation. The messy reality of multivariate stable distributions is that they can have complicated dependence structure and any representation has to be complicated to describe all possibilities. Section 1.4 gives examples of the parameter functions in specific cases, e. g. when there are independent components, a discrete spectral measure, radially or elliptically contoured distributions and sub-stable case. Understanding those examples helps develop an intuition for what happens in general.

1.2.1 Projection based representation

By Theorem 1.1, every linear combination of a multivariate α-stable vector $\langle \mathbf{u}, \mathbf{X} \rangle$ is univariate α-stable for every $\mathbf{u}$, say $\langle \mathbf{u}, \mathbf{X} \rangle \sim \mathbf{S}(\alpha, \beta(\mathbf{u}), \gamma(\mathbf{u}), \delta(\mathbf{u}); k)$. This motivates the first representation of multivariate stable laws.

Definition 1.3 (Projection representation) $\mathbf{X} \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); k)$ if $\mathbf{X}$ is stable with

$$\langle \mathbf{u}, \mathbf{X} \rangle \sim \mathbf{S}(\alpha, \beta(\mathbf{u}), \gamma(\mathbf{u}), \delta(\mathbf{u}); k)$$

for every nonzero $\mathbf{u} \in \mathbb{R}^d$.

In principle, any parameterization k can be used. When $k = 2$, there is no known expression for the mode $m(\alpha, \beta)$ of a standardized univariate stable law, and numerical evidence implies that the mode in the multivariate case (which is not proven to exist in general) is not a linear function of β, so this makes it inconvenient to use in the multivariate setting. Using $k \geq 3$ is possible, but we will not consider that here. In what follows, we will only use parameterizations $k = 0$ and $k = 1$ in the multivariate setting.

This representation has several advantages. First, it builds on the definition of one dimensional stable distributions to give a quick way of describing all multivariate stable distributions: the joint characteristic function is given by

$$
\begin{aligned}
E\exp(i\langle \mathbf{u}, \mathbf{X}\rangle) &= \phi(1|\alpha, \beta(\mathbf{u}), \gamma(\mathbf{u}), \delta(\mathbf{u}); k) \\
&= \begin{cases}
\exp\left(-\gamma^{\alpha}(\mathbf{u})\left[1 + i\tan\frac{\pi\alpha}{2}\beta(\mathbf{u})(\gamma(\mathbf{u})^{1-\alpha} - 1)\right] + i\delta(\mathbf{u})\right) & \alpha \neq 1 \text{ and } k = 0 \\
\exp\left(-\gamma(\mathbf{u})\left[1 + i\frac{2}{\pi}\beta(\mathbf{u})\log\gamma(\mathbf{u})\right] + i\delta(\mathbf{u})\right) & \alpha = 1 \text{ and } k = 0 \\
\exp\left(-\gamma^{\alpha}(\mathbf{u})\left[1 - i\tan\frac{\pi\alpha}{2}\beta(\mathbf{u})\right] + i\delta(\mathbf{u})\right) & \alpha \neq 1 \text{ and } k = 1 \\
\exp\left(-\gamma(\mathbf{u})\left[1 + i\frac{2}{\pi}\beta(\mathbf{u})\log\gamma(\mathbf{u})\right] + i\delta(\mathbf{u})\right) & \alpha = 1 \text{ and } k = 1
\end{cases} \\
&= \begin{cases}
\exp\left(-\gamma^{\alpha}(\mathbf{u}) + i\tan\frac{\pi\alpha}{2}\beta(\mathbf{u})(\gamma(\mathbf{u})^{\alpha} - \gamma(\mathbf{u})) + i\delta(\mathbf{u})\right) & \alpha \neq 1 \text{ and } k = 0 \\
\exp\left(-\gamma(\mathbf{u}) + i\frac{2}{\pi}\beta(\mathbf{u})\gamma(\mathbf{u})\log\gamma(\mathbf{u}) + i\delta(\mathbf{u})\right) & \alpha = 1 \text{ and } k = 0 \\
\exp\left(-\gamma^{\alpha}(\mathbf{u}) - i\tan\frac{\pi\alpha}{2}\beta(\mathbf{u})\gamma^{\alpha}(\mathbf{u}) + i\delta(\mathbf{u})\right) & \alpha \neq 1 \text{ and } k = 1 \\
\exp\left(-\gamma(\mathbf{u}) + i\frac{2}{\pi}\beta(\mathbf{u})\gamma(\mathbf{u})\log\gamma(\mathbf{u}) + i\delta(\mathbf{u})\right) & \alpha = 1 \text{ and } k = 1
\end{cases}
\end{aligned}
$$

where $\phi(\cdot|\alpha, \beta, \gamma, \delta; k)$ is the characteristic function of a one dimensional $\mathbf{S}(\alpha, \beta, \gamma, \delta; k)$ law. Thus specifying the skewness, scale and location parameter for every $\mathbf{u} \in \mathbb{R}^d$ specifies the characteristic function of $\langle \mathbf{u}, \mathbf{X}\rangle$ for every $\mathbf{u}$, i.e. the joint characteristic function. Hence the distribution of the random vector $\mathbf{X}$ is completely determined by the index of stability α, the *skewness function* $\beta(\cdot)$, the *scale function* $\gamma(\cdot)$, and the *location function* $\delta(\cdot)$ on $\mathbb{R}^d$.

A second advantage of this representation is that in Chapter 2 it will be used in formulas for multivariate stable densities and probabilities. Chapter 3 shows that the projection based description is useful when quantifying the closeness of two multivariate stable laws. Finally, Chapter 4 shows that it gives a useful way to estimate multivariate stable distributions by using a sequence of one dimensional stable projections.

Proposition I.1.5 and Theorem 1.1(c) implies the following conditions for strict stability for the projection representation.

Corollary 1.1 $\mathbf{X} \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); 1)$ *is strictly stable if and only if*

$$
\begin{cases}
\delta(\mathbf{u}) = 0 & \alpha \neq 1 \\
\beta(\mathbf{u}) = 0 & \alpha = 1
\end{cases}
$$

for every nonzero $\mathbf{u}$*. In particular, every symmetric stable random vector* $\mathbf{X} \sim \mathbf{S}(\alpha, 0, \gamma(\cdot), 0; 1)$ *is strictly stable.*

When $\alpha = 1$, it is possible for $\beta(\cdot) = 0$ without the spectral measure being symmetric, e.g. Problem 1.1. This means that, unlike the univariate case, there are strictly 1-stable random vectors that are not symmetric around any point.

The functions $\beta(\cdot)$, $\gamma(\cdot)$ and $\delta(\cdot)$ must satisfy certain regularity conditions. Let $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, $\mathbf{e}_d = (0, 0, \ldots, 1)$ be the standard unit basis vectors in $\mathbb{R}^d$ and let $\mathbb{S}$ be the unit sphere in $\mathbb{R}^d$: $\mathbb{S} = \{\mathbf{u} \in \mathbb{R}^d : |\mathbf{u}| = 1\}$. $\mathbb{S}$ is a subset of $\mathbb{R}^d$, but topologically it is a $(d-1)$ dimensional surface. For most of what follows, the dimension d will be fixed and just the symbol $\mathbb{S}$ will be used. In a few cases, there is a need to refer to a two spheres of different dimensions; in those cases the notation $\mathbb{S}_d$ and $\mathbb{S}_m$ will be used to distinguish between the unit spheres in d-dimensions and in m-dimensions. Note that when $d = 1$, the sphere is the two point set $\{-1, +1\} \subset \mathbb{R}$. We will sometimes use geometric reasoning, viewing $\mathbf{u} \in \mathbb{S}$ as a direction in $\mathbb{R}^d$ and $\langle \mathbf{u}, \mathbf{X} \rangle$ is the projection of $\mathbf{X}$ in the direction $\mathbf{u}$.

The scaling properties of $\beta(\cdot)$, $\gamma(\cdot)$ and $\delta(\cdot)$ given below show that they are determined completely by their values on the sphere $\mathbb{S}$. This fact is used repeatedly in the following chapters. In particular, in estimation for multivariate stable laws, one only needs to estimate the parameter functions in for $\mathbf{u} \in \mathbb{S}$, not for every $\mathbf{u} \in \mathbb{R}^d$.

Proposition 1.1 *Let* $\mathbf{X}$ *be stable with* $\langle \mathbf{u}, \mathbf{X} \rangle \sim \mathbf{S}(\alpha, \beta(\mathbf{u}), \gamma(\mathbf{u}), \delta(\mathbf{u}); k)$*,* $k = 0$ *or 1, and let* $c \neq 0$*.*
(a) $\beta(\cdot)$ *is continuous and* $\beta(c\mathbf{u}) = (\operatorname{sign} c)\beta(\mathbf{u})$ *on the set* $\{\mathbf{u} : \gamma(\mathbf{u}) > 0\} \subset \mathbb{R}^d$*. In particular,* $\beta(-\mathbf{u}) = -\beta(\mathbf{u})$*.*
(b) $\gamma(\cdot)$ *is continuous on* $\mathbb{R}^d$*,* $\gamma(\mathbf{u}) \geq 0$*, and* $\gamma(c\mathbf{u}) = |c|\gamma(\mathbf{u})$*. In particular,* $\gamma(-\mathbf{u}) = \gamma(\mathbf{u})$*.*
(c) $\delta(\cdot)$ *is continuous on* $\mathbb{R}^d$ *and*

$$\delta(c\mathbf{u}) = \begin{cases} c\delta(\mathbf{u}) - \frac{2}{\pi} c \log|c| \langle \mathbf{u}, \mathbf{b} \rangle & \alpha = 1, k = 1 \\ c\delta(\mathbf{u}) & \text{otherwise,} \end{cases}$$

where $\mathbf{b} = (\beta(\mathbf{e}_1)\gamma(\mathbf{e}_1), \ldots, \beta(\mathbf{e}_d)\gamma(\mathbf{e}_d))^T$*. When* $\alpha \neq 1$*, there exists a* $\boldsymbol{\delta}_k \in \mathbb{R}^d$ *with* $\delta(\mathbf{u}) = \langle \mathbf{u}, \boldsymbol{\delta}_k \rangle$*.*

There is an ambiguity to the definition of $\beta(\mathbf{u})$ when $\gamma(\mathbf{u}) = 0$. One part of this is a detail: when $\mathbf{u} = 0$, it is unclear how to define $\beta(\mathbf{0})$, which is why we restricted to $\mathbf{u} \neq 0$ above. But it is also possible for $\gamma(\mathbf{u}) = 0$ when $\mathbf{u} \neq 0$. This happens when $\langle \mathbf{u}, \mathbf{X} \rangle$ is degenerate, i.e. $\mathbf{X}$ is supported on a lower dimensional subset, see Theorem 1.4. Problem 1.3 shows examples where there is an essential discontinuity in $\beta(\cdot)$.

1.2.2 Spectral measure representation

While the projection representation has some advantages, it is sometimes cumbersome to work simultaneously with the three functions $\beta(\cdot)$, $\gamma(\cdot)$ and $\delta(\cdot)$. It can be mathematically simpler to use a description of the distribution that involves a measure on the unit sphere. This measure is the method used most often in the literature to describe the joint characteristic function.

To motivate the use of a measure, return to Example 1.1, i.e. $\mathbf{X} = \sum_{j=1}^{n} X_j \mathbf{t}_j$ where each X_j is univariate α-stable and $\mathbf{t}_j \in \mathbb{R}^d$. If some $\mathbf{t}_j = 0$, then it can be eliminated. Also, if $\mathbf{t}_j$ and $\mathbf{t}_k$ are collinear, then we can combine them into one term using the fact that the sum $X_j + X_k$ is α-stable. From now on, we assume $|\mathbf{t}_j| > 0$ for all j and that no two $\mathbf{t}_j$'s are collinear.

Let $X_j \sim \mathbf{S}\left(\alpha, \beta_j, \gamma_j, \delta_j; 1\right)$. Then by Lemma I.3.18, $X_j \stackrel{d}{=} a_j Z_j - a_{j+n} Z_{j+n} + d_j$, where $a_j = ((1+\beta_j)/2)^{-1/\alpha}\gamma_j$, $a_{j+n} = ((1-\beta_j)/2)^{-1/\alpha}\gamma_j$, and some d_j and Z_j, Z_{j+n} are independent, totally right skewed, i.e. $Z_j \sim \mathbf{S}(\alpha, 1; 1)$. If we further write $b_j = b_{j+n} = |\mathbf{t}_j|$ and $\mathbf{s}_j = \mathbf{s}_{j+n} = \mathbf{t}_j/|\mathbf{t}_j|$, then $X_j \mathbf{t}_j = a_j b_j Z_j \mathbf{s}_j - a_j b_j Z_{j+n} \mathbf{s}_j + b_j d_j \mathbf{s}_j$. Hence

$$\mathbf{X} \stackrel{d}{=} \left(\sum_{j=1}^{m} \lambda_j^{1/\alpha} Z_j \mathbf{s}_j \right) + \mathbf{d}, \tag{1.6}$$

where $m = 2n$, $\lambda_j^{1/\alpha} = a_j b_j \geq 0$, $\mathbf{d} = \sum b_j d_j \mathbf{s}_j$. (If any of the original X_j's are totally skewed, i.e. $\beta_j = \pm 1$, then some of the terms will be zero, so the sum may have less than $2n$ terms.)

Equation (1.6) expresses $\mathbf{X}$ as a sum of vector terms, one is a non-random shift and every other term is a one dimensional maximally skewed stable $\lambda_j^{1/\alpha} Z_j$ times a unit vector $\mathbf{s}_j$.

Next, we find the characteristic function of (1.6). Recall that $Z_j \sim \mathbf{S}(\alpha, 1; 1)$ has characteristic function $E \exp(iuZ_j) = \exp(-\omega(u|\alpha, 1; 1))$, where $\omega(u|\alpha, 1; 1)$ is defined on page I.61. By independence of the Z_j's,

$$
\begin{aligned}
E\exp(i\langle \mathbf{u}, \mathbf{X}\rangle) &= E\exp\left(i\sum_j \lambda_j^{1/\alpha} Z_j \langle \mathbf{u}, \mathbf{s}_j\rangle + i\langle \mathbf{u}, \mathbf{d}\rangle\right) \\
&= \left[\prod_j E\exp(i\lambda_j^{1/\alpha} Z_j \langle \mathbf{u}, \mathbf{s}_j\rangle)\right] \exp(i\langle \mathbf{u}, \mathbf{d}\rangle) \\
&= \left[\prod_j \exp(-\omega(\lambda_j^{1/\alpha}\langle \mathbf{u}, \mathbf{s}_j\rangle|\alpha, 1; 1)\right] \exp(i\langle \mathbf{u}, \mathbf{d}\rangle) \\
&= \exp\left(-\sum_j \omega(\lambda_j^{1/\alpha}\langle \mathbf{u}, \mathbf{s}_j\rangle|\alpha, 1; 1) + i\langle \mathbf{u}, \mathbf{d}\rangle\right) \\
&= \exp\left(-\sum_j \omega(\langle \mathbf{u}, \mathbf{s}_j\rangle|\alpha, 1; 1)\lambda_j + i\langle \mathbf{u}, \boldsymbol{\delta}\rangle\right), \qquad (1.7)
\end{aligned}
$$

where the last step uses the scaling property of $\omega(\cdot|\alpha, 1; 1)$ given in Lemma I.3.7 and

$$
\boldsymbol{\delta} = \begin{cases} \mathbf{d} & \alpha \neq 1 \\ \mathbf{d} + \frac{2}{\pi}\sum_j (\lambda_j \log \lambda_j)\,\mathbf{s}_j & \alpha = 1. \end{cases}
$$

The sum in (1.6) with an infinite number of terms ($m = \infty$) will converge in distribution if and only if $\sum \lambda_j < \infty$, because the characteristic function in (1.7) converges pointwise if and only if $\sum \lambda_j < \infty$.

To write this in a different way, let Λ be the discrete measure on $\mathbb{S}$ concentrated on the points $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m\}$, with mass $\Lambda(\{\mathbf{s}_j\}) = \lambda_j$. Here and below, the symbol $\mathbf{1}_{\{\mathbf{s}\in\cdot\}}$ will stand for a unit point measure (a Dirac delta measure) at the point $\mathbf{s}$, i.e. $\mathbf{1}_{\{\mathbf{s}\in A\}} = 1$ if $\mathbf{s} \in A$ and $\mathbf{1}_{\{\mathbf{s}\in A\}} = 0$ if $\mathbf{s} \notin A$. In symbols, the discrete measure can be written as $\Lambda(\cdot) = \sum_j \lambda_j \mathbf{1}_{\{\mathbf{s}_j\in\cdot\}}$. Then (1.7) can be written

$$
E\exp(i\langle \mathbf{u}, \mathbf{X}\rangle) = \exp\left(-\int_{\mathbb{S}} \omega(\langle \mathbf{u}, \mathbf{s}\rangle|\alpha, 1; 1)\Lambda(d\mathbf{s}) + i\langle \mathbf{u}, \boldsymbol{\delta}\rangle\right). \qquad (1.8)
$$

This expresses the log of the characteristic function of $\mathbf{X}$ as an integral of $\omega(\langle \mathbf{u}, \mathbf{s}\rangle|\alpha, 1; 1)$ with respect to a finite discrete measure Λ. This measure Λ is called the *spectral measure* or the *angular measure*.

The next result says that any stable vector can be represented this way if we allow a general measure Λ; it is due to Feldheim (1937). A modern proof of this can be found in Sato (1999), Section 14.

Theorem 1.2 $\mathbf{X}$ *is α-stable if and only if there exists a finite measure Λ on $\mathbb{S}$ and a vector $\boldsymbol{\delta}$ with the characteristic function of $\mathbf{X}$ given by (1.8). When $\alpha < 2$, the pair $(\Lambda, \boldsymbol{\delta})$ is unique.*

When $\alpha = 2$, there are multiple spectral measures that yield the same Gaussian measure, see Problem 1.4.

While the measure Λ completely determines the joint distribution, the way in which it does this is not obvious. One place where the spectral measure does give a

direct meaning is in Theorem 1.5, where it is shown that tail probabilities in cones are determined by the mass the spectral measure assigns to that cone.

The same approach can be followed using any other parameterization of univariate stable: let $Z \sim \mathbf{S}(\alpha, 1; k)$, with $E\exp(iuZ) = \exp(-\omega(u|\alpha, 1; k))$. This leads to the second way of describing multivariate stable distributions in terms of the measure Λ.

Definition 1.4 (Spectral measure representation) $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; k)$, ($k = 0$ or $k = 1$) if

$$E\exp(i\langle \mathbf{u}, \mathbf{X}\rangle) = \exp\left(-\int_{\mathbb{S}} \omega(\langle \mathbf{u}, \mathbf{s}\rangle|\alpha, 1; k)\Lambda(d\mathbf{s}) + i\langle \mathbf{u}, \boldsymbol{\delta}\rangle\right). \tag{1.9}$$

The projection parameter functions from the previous section can be expressed in terms of the spectral measures in the following way.

Theorem 1.3 *(a) Let* $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; 1)$*, then* $\mathbf{X} \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); 1)$*, where*

$$\begin{aligned}
\gamma^\alpha(\mathbf{u}) &= \int_{\mathbb{S}} |\langle \mathbf{u}, \mathbf{s}\rangle|^\alpha \Lambda(d\mathbf{s}) \\
\beta(\mathbf{u}) &= \int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>} \Lambda(d\mathbf{s})/\gamma^\alpha(\mathbf{u}) \text{ when } \gamma(\mathbf{u}) > 0 \\
\delta(\mathbf{u}) &= \begin{cases} \langle \mathbf{u}, \boldsymbol{\delta}\rangle & \alpha \neq 1 \\ \langle \mathbf{u}, \boldsymbol{\delta}\rangle - \frac{2}{\pi}\int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle \log|\langle \mathbf{u}, \mathbf{s}\rangle|\Lambda(d\mathbf{s}) & \alpha = 1. \end{cases}
\end{aligned}$$

(b) Let $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; 0)$*, then* $\mathbf{X} \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); 0)$*, where* $\gamma(\mathbf{u})$ *and* $\beta(\mathbf{u})$ *are the same as in (a) and*

$$\delta(\mathbf{u}) = \begin{cases} \langle \mathbf{u}, \boldsymbol{\delta}\rangle + \tan\frac{\pi\alpha}{2}\left(\beta(\mathbf{u})\gamma(\mathbf{u}) - \langle \mathbf{u}, \boldsymbol{\mu}_\Lambda\rangle\right) & \alpha \neq 1 \\ \langle \mathbf{u}, \boldsymbol{\delta}\rangle + \frac{2}{\pi}\beta(\mathbf{u})\gamma(\mathbf{u})\log\gamma(\mathbf{u}) - \frac{2}{\pi}\int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle \log|\langle \mathbf{u}, \mathbf{s}\rangle|\Lambda(d\mathbf{s}) & \alpha = 1, \end{cases}$$

where

$$\boldsymbol{\mu}_\Lambda = \int_{\mathbb{S}} \mathbf{s}\,\Lambda(d\mathbf{s}) = \left(\int_{\mathbb{S}} s_1\Lambda(d\mathbf{s}), \int_{\mathbb{S}} s_2\Lambda(d\mathbf{s}), \ldots, \int_{\mathbb{S}} s_d\Lambda(d\mathbf{s})\right)^T.$$

(c) Furthermore if $\mathbf{X}$ *is parameterized by* $\mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}_0; 0)$ *and simultaneously by* $\mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}_1; 1)$ *then the location parameters are related by*

$$\boldsymbol{\delta}_1 = \begin{cases} \boldsymbol{\delta}_0 - \tan\frac{\pi\alpha}{2}\boldsymbol{\mu}_\Lambda & \alpha \neq 1 \\ \boldsymbol{\delta}_0 & \alpha = 1 \end{cases}$$

When $\gamma(\mathbf{u}) = 0$, the above definition leaves $\beta(\mathbf{u})$ undefined; recall the discussion after Proposition 1.1. Problem 1.5 gives some properties of $\boldsymbol{\mu}_\Lambda$ and Figure 1.7 shows the effect of the shift in the 0- and 1-parameterizations.

An important special case of the above theorem is when Λ is a symmetric measure on $\mathbb{S}$, e.g. $\Lambda(B) = \Lambda(-B)$ for any Borel set $B \subset \mathbb{S}$, and the shift vector is zero. In this case, $\gamma(\cdot)$ is as above, and $\beta(\cdot) = 0$, $\delta(\cdot) = 0$, $\boldsymbol{\mu}_\Lambda = \mathbf{0}$ and $\boldsymbol{\delta}_0 = \boldsymbol{\delta}_1 = \mathbf{0}$. The joint characteristic function is

$$\phi(\mathbf{u}) = \exp\left(-\gamma^\alpha(\mathbf{u})\right) = \exp\left(-\int_{\mathbb{S}} |\langle \mathbf{u}, \mathbf{s}\rangle|^\alpha \Lambda(d\mathbf{s})\right),$$

and the random vector $\mathbf{X}$ is symmetric: for any Borel set $C \subset \mathbb{R}^d$, $P(\mathbf{X} \in C) = P(\mathbf{X} \in -C)$. Note that this kind of symmetry includes radial symmetry/isotropic and elliptically contoured distributions, but it also includes more general shaped distributions; see the examples in Section 1.4.

Figure 1.7 demonstrates that like the univariate case, the 1-parameterization has a discontinuity in the behavior of the distribution near $\alpha = 1$. The 0-parameterization avoids this discontinuity. As in the univariate case, it is a generally a little more compact and simpler to prove properties using the 1-parameterization; but for numerical computations, estimation, and uniform statements about closeness of distributions with similar spectral measures or projection functions, the 0-parameterization should be used.

The above result gives explicit expressions for the projection parameter functions in terms of the the spectral measure Λ. Conversely, the projection parameter functions determine Λ, but in general there apparently is no simple formula for this. The next chapter details an approximation to the spectral measure using the parameter functions, in Chapter 4 this gives a way to estimate Λ given estimates of the univariate stable parameters of projections $\langle \mathbf{u}, \mathbf{X}\rangle$.

TEXT INCOMPLETE/HIDDEN

Strict stability in the spectral measure representation is characterized by the following.

Corollary 1.2 $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; 1)$ *is strictly stable if and only if*

$$\begin{cases} \boldsymbol{\delta} = 0 & \alpha \neq 1 \\ \boldsymbol{\mu}_\Lambda = \mathbf{0} & \alpha = 1. \end{cases}$$

We saw above that a discrete spectral measure corresponds to a stable distribution that can be written in form (1.6). For practical applications it is easy to work with stable distributions of this type. It turns out that this class is dense in that any stable $\mathbf{Y}$ can be approximated by a stable $\mathbf{X}$ of form (1.6). The precise meaning of this approximation is described in Section 3.6. We will also see that the class of stable random vectors with discrete spectral measures occur in certain types of stable times series and are easy to simulate using (1.6).

Sometimes it is useful to replace the Euclidean sphere $\mathbb{S}$ with some other unit sphere, for example using the unit sphere in the ℓ^p norm on $\mathbb{R}^d$. If $\|\cdot\|$ is another norm on $\mathbb{R}^d$, then define $\mathbb{S}_{\|\cdot\|} = \{\mathbf{s} \in \mathbb{R}^d : \|\mathbf{s}\| = 1\}$ to be the unit sphere in the new norm. Then it is possible to express the joint characteristic function of any stable

random vector $\mathbf{X}$ as

$$E\exp(i\langle \mathbf{u}, \mathbf{X}\rangle) = \exp\left(-\int_{\mathbb{S}_{\|\cdot\|}} \omega(\langle \mathbf{u}, \mathbf{s}\rangle|\alpha, 1; k)\Lambda_{\|\cdot\|}(d\mathbf{s}) + i\langle \mathbf{u}, \boldsymbol{\delta}\rangle\right)$$

as in (1.9). The connection between Λ and $\Lambda_{\|\cdot\|}$ is discussed in Proposition 2.3.8 of Samorodnitsky and Taqqu (1994).

1.2.3 Stable stochastic integral representation

A third way to describe multivariate stable distributions is through a stochastic integral representation. A full definition of the stochastic integrals discussed below is given in Section 3.2 of Samorodnitsky and Taqqu (1994). For here, we will give an informal description, emphasizing the fact that a set of functions $f_1(\cdot), \ldots, f_d(\cdot)$ will determine the joint distribution of a random vector $\mathbf{X}$.

Let $\alpha \in (0, 2]$, $(V, \mathcal{V}, m)$ be a measure space and $\beta^* : V \to [-1, 1]$. Define the classes of $\mathcal{V}$-measurable functions $f : V \to \mathbb{R}$:

$$F = \begin{cases} \{f : \int_V |f(v)|^\alpha m(dv) < \infty\} & \alpha \neq 1 \\ \{f : \int_V |f(v)|m(dv) < \infty \text{ and } \int_V |f(v)\beta^*(v)\log|f(v)||m(dv) < \infty\} & \alpha = 1. \end{cases} \tag{1.10}$$

Define the stable motion $W(t), t \in V$, with control measure m and skewness intensity $\beta^*(\cdot)$ to be the process with stationary independent increments that are stable, with increments $W(t)-W(s) \sim \mathbf{S}\left(\alpha, \beta = \int_{(s,t]} \beta^*(v)m(dv)/m((s,t]), \gamma = (m((s,t]))^{1/\alpha}, 0; 1\right)$. For any $f \in F$, we define the univariate random variable using the symbols

$$X = \int_V f(v)\, dW(v) = \int_V f\, dW$$

to be a stable random variable with distribution $\mathbf{S}\left(\alpha, \beta_g, \gamma_f, \delta_f; 1\right)$, where

$$\gamma_f^\alpha = \int_V |f(v)|^\alpha m(dv)$$

$$\beta_f = \int_V f(v)^{<\alpha>}\beta^*(v)m(dv)/\gamma_f^\alpha \text{ when } \gamma_f > 0$$

$$\delta_f = \begin{cases} 0 & \alpha \neq 1 \\ -\frac{2}{\pi}\int_V f(v)\beta^*(v)\log|f(v)|\, m(dv) & \alpha = 1. \end{cases}$$

Likewise, given d functions $f_1, f_2, \ldots, f_d \in F$, define the d-dimensional random vector

$$\mathbf{X} = \left(\int_V f_1\, dW, \int_V f_2\, dW, \ldots, \int_V f_d\, dW\right)^T \tag{1.11}$$

to be a stable random vector with distribution $\mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); 1)$, where

$$\gamma^\alpha(\mathbf{u}) = \int_V \left| \sum_{j=1}^d u_j f_j(v) \right|^\alpha m(dv) \tag{1.12}$$

$$\beta(\mathbf{u}) = \int_V \left(\sum_{j=1}^d u_j f_j(v) \right)^{<\alpha>} \beta^*(v) m(dv) / \gamma^\alpha(\mathbf{u}) \text{ when } \gamma(\mathbf{u}) > 0$$

$$\delta(\mathbf{u}) = \begin{cases} 0 & \alpha \neq 1 \\ -\frac{2}{\pi} \int_V \left(\sum_{j=1}^d u_j f_j(v) \right) \beta^*(v) \log \left| \sum_{j=1}^d u_j f_j(v) \right| m(dv) & \alpha = 1. \end{cases}$$

An important special case is when the skewness intensity $\beta^*(v) = 0$ for all v. In this case, the random vector (1.11) is symmetric with distribution $\mathbf{S}(\alpha, 0, \gamma(\cdot), 0; 1)$, where $\gamma(\mathbf{u})$ is given by (1.12).

The main point here is that the family $f_1, \ldots, f_d$ of functions in F, along with the skewness intensity $\beta^*(\cdot)$ and control measure m, determine the joint distribution of $\mathbf{X}$. We will see that this construction gives a well defined stochastic integral based on stable distributions. One important feature of this stochastic integral is that it generalizes to stable processes $X(t) = \int_V f_t(v) dW(v)$ on an interval $t \in [0, T]$ or more general index set, where $\{f_t(\cdot)\} \subset F$.

1.2.4 Stochastic series representation

Section I.3.12 described how a univariate stable random variable can be represented by a series. The next representation generalizes that approach to random vectors as follows. Let $0 < \alpha < 2$, $(V, \mathcal{V}, m)$ be a finite measure space, $\beta^*(\cdot)$ a skewness intensity on V and F be defined by (1.10). Let $Z_1, Z_2, \ldots$ be a sequence of independent V-valued random variables with distribution $\widehat{m}(\cdot) := m(\cdot)/m(V)$. Further, define a sequence $\epsilon_1, \epsilon_2, \ldots$ of ± 1 valued random variables that are independent of each other but with ϵ_j dependent on Z_j in the following way:

$$P(\epsilon_j = +1 | Z_j) = 1 - P(\epsilon_j = -1 | Z_j) = \frac{1 + \beta^*(Z_j)}{2}.$$

Further, let $G_j = E_1 + \cdots + E_j$, where $E_1, E_2, \ldots$ are i.i.d. r.v. with a standard exponential distribution. Then for any $f_1, \ldots, f_d \in F$, consider

$$\mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix} = a \sum_{j=1}^\infty \left[\epsilon_j G_j^{1/\alpha} \right] \begin{pmatrix} f_1(Z_j) \\ \vdots \\ f_d(Z_j) \end{pmatrix} - \begin{pmatrix} b_{j1} \\ \vdots \\ b_{jd} \end{pmatrix} + \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix} \tag{1.13}$$

where

$$a = \left(\tfrac{2}{\pi}\Gamma(\alpha)\sin\tfrac{\pi\alpha}{2}\, m(V)\right)^{1/\alpha}$$

$$b_{jk} = \begin{cases} 0 & 0 < \alpha < 1 \\ \left(\int_{1/j}^{1/(j-1)} x^2 \sin x\, dx\right) \int_V f_k(v)\beta^*(v)\, m(dv) & \alpha = 1 \\ \frac{\alpha}{\alpha-1}\left(j^{(\alpha-1)/\alpha} - (j-1)^{(\alpha-1)\alpha}\right) \int_V f_k(v)\beta^*(v)\, m(dv) & 1 < \alpha < 2 \end{cases}$$

$$c_k = \begin{cases} 0 & \alpha \neq 1 \\ \frac{2}{\pi}\log(\frac{2}{\pi}m(V)) \int_V f_k(v)\beta^*(v) m(dv) & \alpha = 1 \end{cases}$$

Section 3.10 of Samorodnitsky and Taqqu (1994) shows that the series above converges and that the result is $\mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); 1)$ where $\beta(\cdot)$, $\gamma(\cdot)$ and $\delta(\cdot)$ are given by (1.12).

When $\beta^*(\cdot) = 0$, the series (1.13) simplifies to

$$\mathbf{X} = a \sum_{j=1}^{\infty} \epsilon_j G_j^{1/\alpha} \begin{pmatrix} f_1(Z_j) \\ \vdots \\ f_d(Z_j) \end{pmatrix},$$

where ϵ_j is ± 1 with probability 1/2 and independent of Z_j, and the result is symmetric stable.

1.2.5 Zonoid representation

It is possible to give a geometric description of symmetric multivariate stable laws using *zonoids*. These are sets in $\mathbb{R}^d$ with certain additional properties, see Schneider (1993). Let $\mathbf{X} = (X_1, \ldots, X_d)^T$ be a symmetric α-stable random vector, e.g. $\mathbf{X} \sim \mathbf{S}(\alpha, 0, \gamma(\cdot), 0; 1)$. Define the set $K = \{\mathbf{u} \in \mathbb{R}^d : \gamma(\mathbf{u}) \leq 1\}$. K can be thought of as the unit ball in the norm $\|\mathbf{u}\|_{\mathbf{X}} = \gamma(\mathbf{u})$. This is a closed symmetric set that contains 0, and when $\alpha \geq 1$ the triangle inequality shows that K is a convex set. In addition to being convex, the set K has a restricted geometry, specifically the set must be an L^α-zonoid, see Molchanov (2009) for definitions and proofs. From a probabilistic viewpoint, this is the geometric condition necessary for $\exp(-\gamma^\alpha(\mathbf{u}))$ to be a characteristic function. Conversely, if K is an L^α-zonoid, then it determines the scale function through the expression $\gamma^\alpha(\mathbf{u}) = h(K, \mathbf{u}) := \sup\{\langle \mathbf{u}, \mathbf{v}\rangle : \mathbf{v} \in K\}$. In analysis, the norm $\|\mathbf{u}\|_{\mathbf{X}}$ is called the gauge function and $h(K, \cdot)$ is called the support function of K. This representation in terms of a convex set allows one to use results in convex analysis to derive results about multivariate stable laws. Here are two examples. If X_1 and X_2 are independent symmetric α-stable random vectors with respective zonoids K_1 and K_2, then the zonoid associated with $X_1 + X_2$ is the Minkowski sum $K_1 + K_2 = \{\mathbf{v}_1 + \mathbf{v}_2 : \mathbf{v}_1 \in K_1, \mathbf{v}_2 \in K_2\}$. And the value of the density of $\mathbf{X}$ at the origin is directly proportional to the volume of K, see Molchanov (2009).

When $\alpha < 1$, the set K may be convex, e.g. when $\mathbf{X}$ is sub-Gaussian, or not, e.g. when $\mathbf{X}$ has independent components. While it is not generally convex, it does

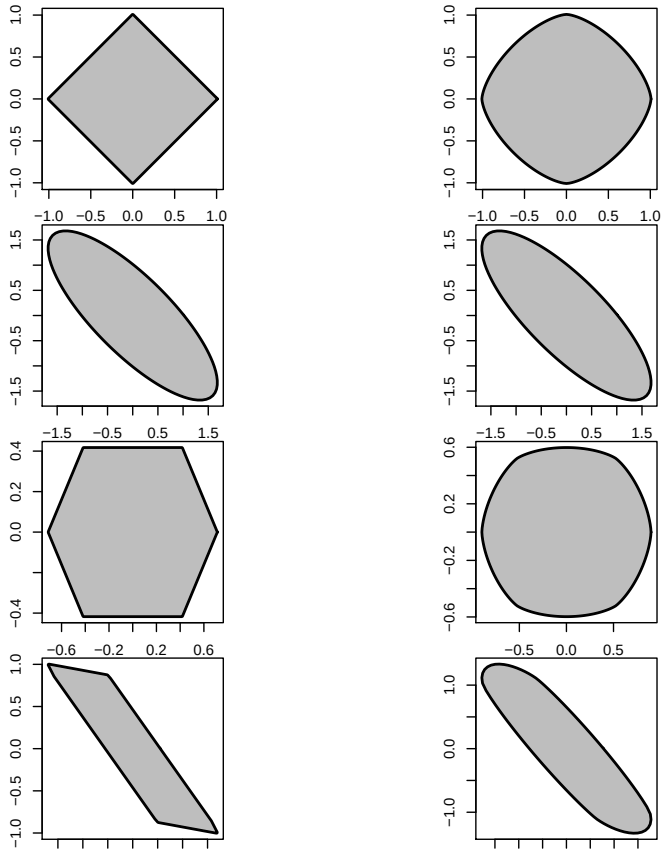

Fig. 1.1 Zonoids for some bivariate symmetric stable laws. The left column is with $\alpha = 1$ and the right column is with $\alpha = 1.5$. The first row is when the components are independent; the second row is when the distribution is elliptically contoured with shape matrix $R = [1, 0.8; 0.8, 1]$; the third row is when the spectral measure is discrete with 6 point masses at anlges θ_i at $\pi/4$, $\pi/2$, $3\pi/4$, $5\pi/4$, $3\pi/2$, $7\pi/4$, each point having mass 1/2; the last row is from a discrete angular measure with 4 point masses: two at angles $\pi/6$ and $7\pi/6$ with mass 1/2, and two at angles $\pi/3$ and $4\pi/3$ with mass 1.

uniquely determine the joint distribution of $\mathbf{X}$. Figure 1.1 shows the zonoids associated with some bivariate symmetric α-stable laws.

1.3 Multivariate stable densities and probabilities

As in the one dimensional case, there are no general formulas for the densities or cumulative d.f. of a multivariate stable distribution. First, there is the question of

when a multivariate stable distribution has a density, i.e. is nonsingular, and what the support of the density is. Lemma I.1.1 shows that the support of a one dimensional stable distribution can be either the whole line or a half-line (when $\alpha < 1$ and $|\beta| = 1$). In the multivariate setting, when a density exists, it can be supported on a proper subset of $\mathbb{R}^d$. For example, $\mathbf{X} = (X_1, 0)^T$ is formally a two dimensional stable distribution if X_1 is univariate stable, but it is supported on the one dimensional subspace $\{(x, 0) : x \in \mathbb{R}\}$. Note that $\gamma(\mathbf{e}_2) =$ the scale of $\langle \mathbf{e}_2, \mathbf{X} \rangle = 0 \cdot X_1 + 1 \cdot 0$ is 0 in this case, i.e. there is no spread in the direction $\mathbf{e}_2$. The Corollary below shows that if $\gamma(\mathbf{u}) > 0$ for all $\mathbf{u} \in \mathbb{S}$, then $\mathbf{X}$ has a density.

In addition being supported on a lower dimensional subspace, the support can be a cone. To specify the support of a multivariate stable distribution, we need some definitions. First, the support of a measure, e.g. support (Λ) is the largest closed set for which every neighborhood of any point in the set has positive measure. We will also use the notation support $(\mathbf{X})$ to denote the support of the associated probability measure of the random vector $\mathbf{X}$. For a subspace B of $\mathbb{R}^d$,

$$B^{\perp} = \{\mathbf{v} \in \mathbb{R}^d : \langle \mathbf{v}, \mathbf{b} \rangle = 0 \text{ for all } \mathbf{b} \in B\}.$$

The linear span of a set $A \subset \mathbb{R}^d$ is

$$\text{span}\,(A) = \{c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n : \mathbf{a}_1, \ldots, \mathbf{a}_n \in A,\ c_1, \ldots, c_n \in \mathbb{R}\}.$$

The closed convex hull (CCH) of A is

$$\text{CCH}(A) = \overline{\{c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n : \mathbf{a}_1, \ldots, \mathbf{a}_n \in A,\ c_1, \ldots, c_n \geq 0,\ \sum c_j = 1\}}.$$

Finally the cone generated by A is

$$\text{Cone}(A) = \{r\mathbf{a} \in \mathbb{R}^d : r > 0, \mathbf{a} \in A\},$$

see Figure 1.2. Note that span (A) includes negative coefficients c_i, but $\text{Cone}(\text{CCH}(A)) = \overline{\{c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n : \mathbf{a}_1, \ldots, \mathbf{a}_n \in A,\ c_1, \ldots, c_n \geq 0\}}$ does not. Thus we always have

$$\text{Cone}(\text{CCH}(A)) \subset \overline{(}\text{span}\,(A)),$$

and the inclusion is strict if and only if A is contained in a half-space (whose boundary goes through the origin). It follows that for any A

$$\text{span}\,(\text{Cone}(\text{CCH}(A)) = \text{span}\,(\text{CCH}(A)) = \overline{(}\text{span}\,(A)). \tag{1.14}$$

The following theorem is from Port and Vitale (1988) for $0 < \alpha \leq 1$, and from Ashbaugh et al. (1992) for $1 < \alpha < 2$. There is an alternative derivation of this in Section 2 of Modarres and Nolan (1994). Further, the support is similar in the infinite dimensional case, see Rajput (1993, 1994).

Theorem 1.4 ***(Support of a multivariate stable law)*** *For* $\mathbf{X} \sim \mathbf{S}\,(\alpha, \Lambda, \boldsymbol{\delta}; 1)$

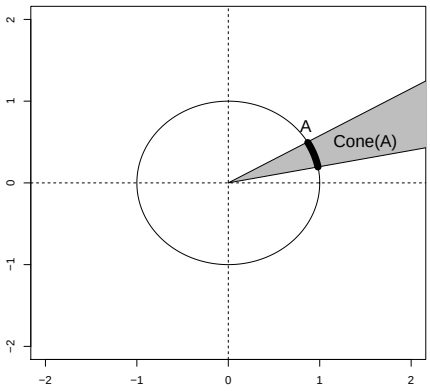

Fig. 1.2 The set Cone(A) in $\mathbb{R}^2$ for $A = \{\mathbf{s} : \mathbf{s} = (\cos\theta, \sin\theta), \pi/16 \le \theta \le \pi/6\}$.

$$\text{support}\,\mathbf{X} = \begin{cases} \text{Cone}(\text{CCH}(\text{support}\,(\Lambda))) + \boldsymbol{\delta} & 0 < \alpha < 1 \\ \text{span}\,(\text{support}\,(\Lambda)) + \boldsymbol{\delta} & 1 \le \alpha \le 2. \end{cases}$$

If $0 < \alpha < 1$ and support Λ is contained in a half-space, then the support of $\mathbf{X}$ will not be all of $\mathbb{R}^d$. A simple two dimensional example is when Λ consists of two point masses: one at $\mathbf{e}_1$ and one at $\mathbf{e}_2$, both with unit mass. The corresponding stable random vector $\mathbf{X}$ has two independent components, each of which is $\mathbf{S}\,(\alpha, 1; 1)$. As $0 < \alpha < 1$, each component is supported on $[0, \infty)$, so the joint density (the product of the marginals) is positive precisely on the interior of the first quadrant. As the previous theorem says, Cone$(CCH(\text{support}\,\Lambda))$ is the first quadrant. See the bottom plot of Figure 1.4 for this example and the bottom plot of Figure 1.5 for one with the cone being bigger. If support Λ is not contained in a half-space, then Cone$(CCH(\text{support}\,\Lambda))$ will be $\mathbb{R}^d$, see the top plot in Figure 1.4, where there are unit point masses at $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. A useful case in dimension $d \ge 2$ is when a spectral measure is supported in the positive orthant, in which case the corresponding stable vector has non-negative components.

The most important case is when $\mathbf{X}$ has full support and it has a density. For a stable random vector $\mathbf{X}$, let $\gamma(\cdot)$ be it's scale function and define

$$\gamma_{\min} = \min_{\mathbf{u} \in \mathbb{S}} \gamma(\mathbf{u}). \tag{1.15}$$

Note that if $\gamma(\mathbf{u}) > 0$ for all $\mathbf{u} \ne 0$, then continuous $\gamma(\mathbf{u})$ achieves a minimum $\gamma_{\min}$ on compact $\mathbb{S}$ and necessarily $\gamma_{\min} > 0$. Conversely, if $\gamma_{\min} > 0$, then the scaling property from Proposition 1.1 implies $\gamma(\mathbf{u}) = |\mathbf{u}|\gamma(\mathbf{u}/|\mathbf{u}|) \ge |\mathbf{u}|\gamma_{\min} > 0$ for $\mathbf{u} \ne 0$.

Corollary 1.3 *Let* $\mathbf{X} \sim \mathbf{S}\,(\alpha, \Lambda, \boldsymbol{\delta}; 1)$ *be a d-dimensional stable distribution with* $\gamma_{\min} > 0$*. Then* $\mathbf{X}$ *has a density* $f(\mathbf{x})$ *on* $\mathbb{R}^d$ *and*

$$\text{support}\,\mathbf{X} = \begin{cases} \text{Cone}(\text{CCH}(\text{support}\,(\Lambda))) + \boldsymbol{\delta} & 0 < \alpha < 1 \\ \mathbb{R}^d & 1 \le \alpha \le 2. \end{cases}$$

The support of a multivariate stable law is always a cone. When $\gamma_{\min} = 0$, the support will be a cone that is supported on a lower dimensional subspace of $\mathbb{R}^d$ as specified in Theorem 1.4. Restricted to that subspace, the support of $\mathbf{X}$ is like the full dimension case above and there is a density on that lower dimensional subspace.

Expressions for the multivariate stable densities $f(\mathbf{x})$ and probabilities $P(\mathbf{X} \in A)$ are given in Section 2.4. Those formulas involve both special functions and multivariate integrals. Programs for computing multivariate stable densities exist for some classes of stable laws in two dimensions; Section 1.4 shows some examples. For higher dimensions, the only cases where there are numerical methods are when (a) there is a discrete spectral measure with exactly d point masses, see (1.21), or (b) the elliptical case, see Section 2.5.

It is conjectured that multivariate stable densities are unimodal. Wolfe (1975) proved that spherically symmetric stable laws are unimodal, but there is no proof known for the general case. The plots shown below have a clear mode, but since there is no closed form expression for most multivariate stable laws and the proof of unimodality in the univariate case is complicated, proving unimodality in the multivariate case is likely to be quite difficult.

1.3.1 Multivariate tail probabilities

Multivariate stable distributions have heavy tails, although the tail behavior generally varies in different directions. For a set $A \subset \mathbb{S}$, ∂A will denote the boundary of A and $\overline{A}$ will denote the closure of A in $\mathbb{S}$. We state the next result and refer the reader to Meerschaert and Scheffler (2001), Section 8.2 for a proof.

Theorem 1.5 ***Multivariate tail probabilities.*** *Let* $\mathbf{X} \sim \mathbf{S}\,(\alpha, \Lambda, \boldsymbol{\delta}; k)$.
(a) As $r \to \infty$

$$r^\alpha P(|\mathbf{X}| > r) \to \frac{1}{\alpha}\Lambda(\mathbb{S}).$$

(b) For any subset of $A \subset \mathbb{S}$ *with* $\Lambda(\partial A) = 0$*, as* $r \to \infty$

$$r^\alpha P(\mathbf{X} \in \text{Cone}(A), |\mathbf{X}| > r) \to \frac{1}{\alpha}\Lambda(A)$$

This generalizes the one dimensional result in Theorem I.1.2, showing an asymptotic power law tail behavior for $|\mathbf{X}|$. The second part of the theorem shows that tail behavior in "direction" A is determined by the mass that Λ assigns to A. As in the one dimensional case, there can be directions where the tail behavior is not Pareto. One example of this is when the support of the distribution is a proper cone as in Theorem 1.4. Even when the support is all of $\mathbb{R}^d$, for any set A which has $\Lambda(\overline{A}) = 0$,

the right hand side of (b) above is zero and directional tail probabilities are lighter than Pareto.

Unlike the univariate case, there is no simple general statement about the directional behavior of multivariate stable densities. Indeed, the tail behavior of multivariate stable densities can be very different along different rays. See Problem 1.6 for a simple example of different tail behavior in different directions and Section 2.4.1 for more discussion.

1.4 Examples of multivariate stable distributions

Since the set of all multivariate stable laws is very large, it is useful to identify several classes that are more concrete and accessible.

1.4.1 Independent components

Let $\mathbf{X}$ have independent components, with the j^{th} component $X_j \sim \mathbf{S}\left(\alpha, \beta_j, \gamma_j, \delta_j; 1\right)$. This happens if and only if the spectral measure is concentrated at the points of intersection of the coordinate axes and the sphere, e.g. on $\{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \ldots, \pm\mathbf{e}_d\}$, where $\mathbf{e}_j$ is the j^{th} standard unit basis vector in $\mathbb{R}^d$. Specifically, the spectral measure is discrete with mass $\frac{1}{2}(1+\beta_j)\gamma_j^\alpha$ at $\mathbf{e}_j$ and mass $\frac{1}{2}(1-\beta_j)\gamma_j^\alpha$ at $-\mathbf{e}_j$. In symbols, the spectral measure and shift vectors are:

$$\Lambda(\cdot) = \sum_{j=1}^{d} \frac{1}{2}(1+\beta_j)\gamma_j^\alpha \mathbf{1}_{\{\mathbf{e}_j \in \cdot\}} + \sum_{j=1}^{d} \frac{1}{2}(1-\beta_j)\gamma_j^\alpha \mathbf{1}_{\{-\mathbf{e}_j \in \cdot\}}, \quad \boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_d)^T.$$

Consequently, by Theorem 1.3 the parameter functions are

$$\begin{aligned}
\gamma^\alpha(\mathbf{u}) &= \sum_{j=1}^{d} |u_j|^\alpha \gamma_j^\alpha \\
\beta(\mathbf{u}) &= \left(\sum_{j=1}^{d} u_j{}^{<\alpha>} \beta_j \gamma_j^\alpha \right) / \gamma^\alpha(\mathbf{u}) \\
\delta(\mathbf{u}) &= \begin{cases} \langle \mathbf{u}, \boldsymbol{\delta} \rangle & \alpha \neq 1 \\ \langle \mathbf{u}, \boldsymbol{\delta} \rangle - \frac{2}{\pi} \sum_{j=1}^{d} u_j \log |u_j| \beta_j \gamma_j^\alpha & \alpha = 1. \end{cases}
\end{aligned}$$

In this case, the joint density factors into a product of marginals:

$$f_{\mathbf{X}}(\mathbf{x}; k) = \prod_{j=1}^{d} f(x_j | \alpha, \beta_j, \gamma_j, \delta_j; k).$$

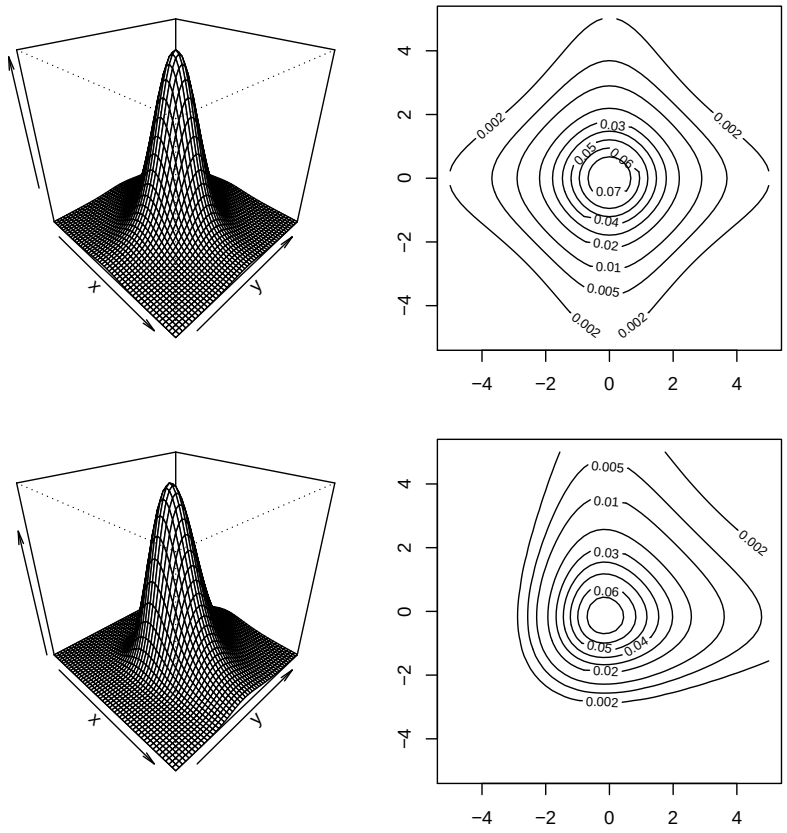

Fig. 1.3 Density surface and contours for independent components with $\alpha = 1.5$. At top, $\beta_1 = \beta_2 = 0$; at bottom $\beta_1 = \beta_2 = 1$.

This can be computed using the numerical methods for univariate stable densities in Chapter I.3. Figures 1.3 and 1.4 show some bivariate density plots.

1.4.2 Discrete spectral measures

If $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; k)$ with the spectral measure discrete, say $\Lambda(\cdot) = \sum_{j=1}^{m} \lambda_j \mathbf{1}_{\{\mathbf{s}_j \in \cdot\}}$, then the parameter functions are easy to describe: by Theorem 1.3

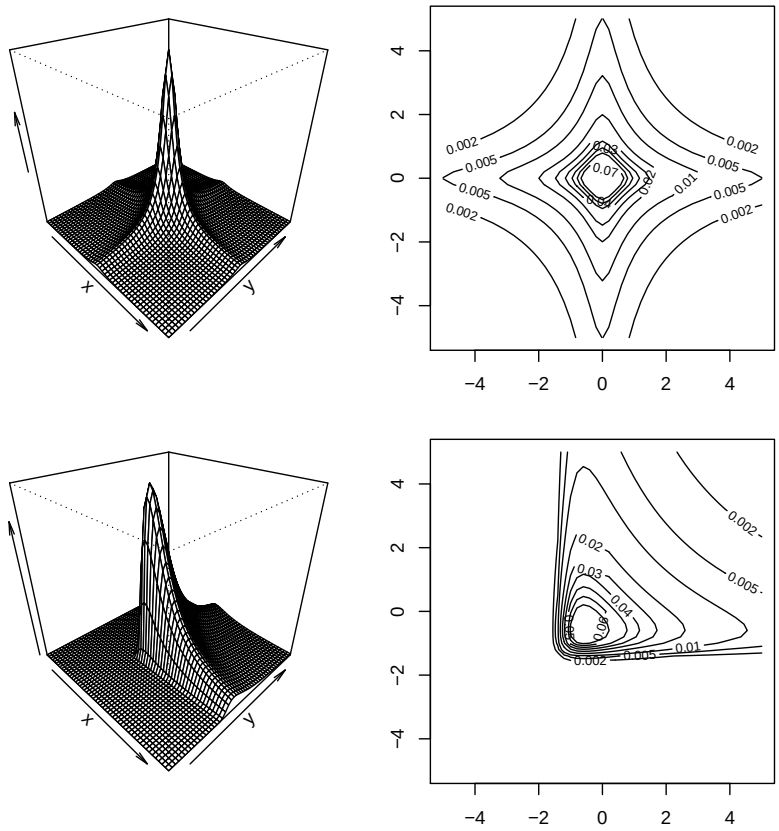

Fig. 1.4 Density surface and contours for independent components with $\alpha = 0.75$. At top, $\beta_1 = \beta_2 = 0$; at bottom $\beta_1 = \beta_2 = 1$. Since $\alpha < 1$, and the support of the spectral measure is in the cone generated by unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, the support is the first quadrant for the bottom example.

$$\gamma^\alpha(\mathbf{u}) = \sum_{j=1}^{m} |\langle \mathbf{u}, \mathbf{s}_j \rangle|^\alpha \lambda_j$$

$$\beta(\mathbf{u}) = \left(\sum_{j=1}^{m} \langle \mathbf{u}, \mathbf{s}_j \rangle^{<\alpha>} \lambda_j \right) / \gamma^\alpha(\mathbf{u})$$

$$\delta(\mathbf{u}) = \begin{cases} \langle \mathbf{u}, \boldsymbol{\delta} \rangle & \alpha \neq 1, k = 1 \\ \langle \mathbf{u}, \boldsymbol{\delta} \rangle - \frac{2}{\pi} \sum_j \langle \mathbf{u}, \mathbf{s}_j \rangle \log |\langle \mathbf{u}, \mathbf{s}_j \rangle| \lambda_j & \alpha = 1, k = 1 \\ \langle \mathbf{u}, \boldsymbol{\delta} \rangle + \tan \frac{\pi\alpha}{2} (\beta(\mathbf{u})\gamma(\mathbf{u}) - \langle \mathbf{u}, \boldsymbol{\mu}_\Lambda \rangle) & \alpha \neq 1, k = 0 \\ \langle \mathbf{u}, \boldsymbol{\delta} \rangle + \frac{2}{\pi} \beta(\mathbf{u})\gamma(\mathbf{u}) \log \gamma(\mathbf{u}) - \frac{2}{\pi} \sum_j \langle \mathbf{u}, \mathbf{s}_j \rangle \log |\langle \mathbf{u}, \mathbf{s}_j \rangle| \lambda_j & \alpha = 1, k = 0 \end{cases}$$

$$\boldsymbol{\mu}_\Lambda = \left(\sum_j s_{j1} \lambda_j, \ldots, \sum_j s_{jd} \lambda_j \right).$$

Section 1.2.2 showed that a discrete spectral measure is equivalent to being able to write $\mathbf{X} = A\mathbf{Z}+\boldsymbol{\delta}$, where A is a $d\times m$ real matrix and $\mathbf{Z} = (Z_1, \ldots, Z_m)^T$ is an m vector of i.i.d. $\mathbf{S}(\alpha, 1; k)$ random variables. If we write $A = \left(\lambda_1^{1/\alpha}\mathbf{s}_1, \lambda_2^{1/\alpha}\mathbf{s}_2, \ldots, \lambda_m^{1/\alpha}\mathbf{s}_m\right)$, i.e. $\mathbf{s}_j = A_{\cdot j}/|A_{\cdot j}|$ is a unit vector in the same direction as the j-th column of A and $\lambda_j = |A_{\cdot j}|^\alpha$, then $\mathbf{X} \stackrel{d}{=} \sum_{j=1}^m \lambda_j^{1/\alpha}\mathbf{s}_j Z_j + \boldsymbol{\delta}$.

It is sometimes useful to allow slightly more generality: consider

$$\mathbf{X} = \sum_{j=1}^{m} w_j \mathbf{t}_j Z_j + \boldsymbol{\delta}, \tag{1.16}$$

where $\mathbf{t}_j$, $j = 1, \ldots, m$ are arbitrary vectors in $\mathbb{R}^d$, $w_1, \ldots, w_m$ are arbitrary real coefficients, $Z_j \sim \mathbf{S}(\alpha, \beta_j, \gamma_j, \delta_j; k)$ are independent, and $\boldsymbol{\delta} \in \mathbb{R}^d$. This happens if and only if the spectral measure is concentrated on the intersection of the sphere with the m lines in the direction of the vectors $\mathbf{t}_1, \ldots, \mathbf{t}_m$. To be precise, the spectral measure has mass

$$\lambda_j = w_j\gamma_j \left(\frac{1+\beta_j}{2}\right)^{1/\alpha} \|\mathbf{t}_j\| \quad \text{at} \quad \mathbf{s}_j = \mathbf{t}_j/\|\mathbf{t}_j\|, \quad j = 1, \ldots, m \tag{1.17}$$

$$\lambda_{j+m} = w_j\gamma_j \left(\frac{1-\beta_j}{2}\right)^{1/\alpha} \|\mathbf{t}_j\| \quad \text{at} \quad \mathbf{s}_{j+m} := -\mathbf{s}_j, \quad j = 1, \ldots, m. \tag{1.18}$$

In matrix notation:

$$\mathbf{X} = A\mathbf{Z} + \boldsymbol{\delta}, \tag{1.19}$$

where A is an $(d \times m)$ matrix and $\mathbf{Z} = (Z_1, \ldots, Z_m)^T$ has independent components $Z_j \sim \mathbf{S}(\alpha, \beta_j, \gamma_j, \delta_j; k)$, and $\boldsymbol{\delta} \in \mathbb{R}^d$. The spectral measure is discrete with the point masses in the directions of the columns of A. In general, the spectral measure for (1.16) or (1.19) will have $2m$ point masses, but it can have less in the following cases:

- If $\beta_j = +1$ for some j, then $\lambda_{j+m} = 0$ by (1.18); if $\beta_j = -1$ for some j, then $\lambda_j = 0$ by (1.17).
- If some $\mathbf{t}_j = \mathbf{0}$ (equivalently column j of A is zero), then $\lambda_j = \lambda_{j+m} = 0$.
- If some of the vectors $\mathbf{t}_j$ in (1.16) (equivalently columns of A in (1.19)), point in the same direction, then the weights in that direction can be combined.

Note that $m = \infty$ is allowed in (1.16) if the sum $\sum_{j=1}^\infty |w_j\mathbf{t}_j|^\alpha$ is finite. Equivalently, A in (1.19) can be a $(d \times \infty)$ matrix with row sums $\sum_{j=1}^\infty |a_{i,j}|^\alpha < \infty$, $i = 1, \ldots, d$.

Suppose $\mathbf{X}$ can be written in form (1.19) where A is a $d \times m$ matrix and $\mathbf{Z}$ has m i.i.d. $\mathbf{S}(\alpha, 1; k)$ components. For B a $p \times d$ matrix, $\mathbf{Y} = B\mathbf{X} = (BA)\mathbf{Z} + (B\boldsymbol{\delta})$, so $\mathbf{Y}$ has a discrete spectral measure with exactly the same number of point masses as $\mathbf{X}$. As discussed above, the point masses of the spectral measure for $\mathbf{X}$ are at $\mathbf{s}_j = A_{\cdot j}/|A_{\cdot j}|$ and have mass $\lambda_j = |A_{\cdot j}|^\alpha$, $j = 1, \ldots, m$. (Here and below, $A_{\cdot j}$ is the $j-$th column of A and $A_{i\cdot}$ is the $i-$th row of A.) For notational convenience, let $C = BA$, a $p \times m$ matrix. The spectral measure for $\mathbf{Y}$ will have point masses λ_j^* at location $\mathbf{s}_j^*$, $j = 1, \ldots, m$ where

$$\mathbf{s}_j^* = \frac{C_{\cdot j}}{|C_{\cdot j}|}, \qquad \lambda_j^* = |C_{\cdot j}|^\alpha \lambda_j. \tag{1.20}$$

The case of independent components was discussed above, where all the mass of Λ is concentrated along the axes. A related special case is when there are exactly d terms, say $\mathbf{Y} = A\mathbf{X}$, where A is a $d \times d$ matrix and $\mathbf{X}$ has independent components. If A is not invertible, then $\mathbf{Y}$ does not have a density on $\mathbb{R}^d$. When A is invertible, the joint density of $\mathbf{Y}$ can also be expressed in terms of univariate stable densities, see Problem 1.7 for a proof of the following. Let $\mathbf{x} = (x_1(\mathbf{y}), \ldots, x_d(\mathbf{y}))^T = A^{-1}\mathbf{y}$, then the density of $\mathbf{Y}$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}; k) = \frac{\prod_{j=1}^d f(x_j(\mathbf{y})|\alpha, \beta_j, \gamma_j, \delta_j; k)}{|\det A|}. \tag{1.21}$$

Discrete spectral measures occur naturally in several contexts. If X_t is a linear process, e.g. $X_t = \sum_k Z_{k-t}$ where Z_k's are independent, then the spectral measure for the finite dimensional distributions of X are discrete. It is shown in Chapter 3 that stable random vectors with discrete spectral measures are dense in the sense that any stable random vector can be approximated by a discrete spectral measure. And the finite dimensional distributions of the stable Ornstein-Uhlenbeck process, see Section 6.10.3, have discrete spectral measures.

We note that when $\mathbf{X}$ is symmetric, we can reduce the number of terms in (1.16) by a half. Rather than use the $\mathbf{S}(\alpha, 1; 1)$ terms there with spectral mass at antipodal points, we can use $m/2$ terms in (1.19) with $\beta_j = 0$. This is computationally efficient if we are computine the projection parameter functions or simulating or computing densities in the symmetric case.

Figure 1.5 shows some examples of densities from discrete spectral measures. Notice how the contour plots show level curves that are roughly elliptical near the origin, but bulge out along the directions where there are point masses for large $|\mathbf{x}|$. If we increase the number of point masses, but keep the relative masses comparable, the contours round out in the region shown; however far from the origin there will still be bulges in the directions of the point masses because of Theorem 1.5.

1.4.3 Radially symmetric and elliptically contoured stable laws

Isotropic or radially symmetric stable distributions arise naturally in some physical problems. Unlike the univariate case, radially symmetric is not the same as symmetric: the former implies uniform behavior in all directions, the latter only implies that $\mathbf{X} \stackrel{d}{=} -\mathbf{X}$. Without loss of generality, we assume the distributions are radially symmetric around the origin; radial symmetry around another point are simple shifts of this case. In this case, expressions simplify: the spectral measure is a constant multiple of surface measure (uniform or Haar measure) on the sphere and the parameter func-

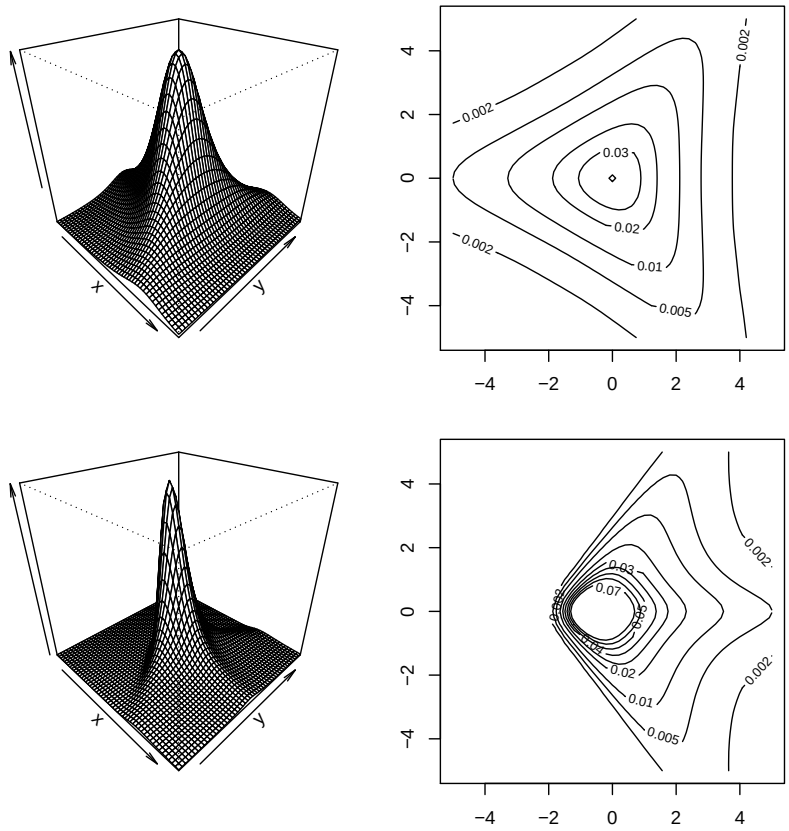

Fig. 1.5 Density surface and contours for dependent components with discrete spectral measure. The top shows $\alpha = 1.5$ with point masses at $s_j = (\cos\theta_j, \sin\theta_j)$, $\theta_1 = \pi/3$, $\theta_2 = \pi$, $\theta_1 = 5\pi/3$, and $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$. The bottom shows $\alpha = 0.8$ with point masses at $s_j = (\cos\theta_j, \sin\theta_j)$, $\theta_1 = 0$, $\theta_2 = \pi/3$, $\theta_1 = 5\pi/3$, and $\lambda_1 = 1/2$, $\lambda_2 = 1/2$, $\lambda_3 = 1/2$.

tions are $\beta(\mathbf{u}) = 0$, $\gamma(\mathbf{u}) = \gamma_0|\mathbf{u}|$ and $\delta(\mathbf{u}) = 0$. Thus there are only three parameters: the index of stability α, a scale parameter γ_0 and the dimension d.

There are two cases where the densities are known in closed form.

Example 1.6 Radially symmetric Gaussian and Cauchy distributions. Let $\gamma_0 > 0$ and dimension $d \geq 1$. Radially symmetric Gaussian distributions ($\alpha = 2$) have densities

$$f(\mathbf{x}) = \frac{1}{(2\pi\gamma_0^2)^{d/2}} \exp(-|\mathbf{x}|^2/(4\gamma_0^2)).$$

Note that there is the usual confusion between the Gaussian scale and the stable scale when $\alpha = 2$: the covariance of the distribution given by $f(\mathbf{x})$ above is $2\gamma_0^2 I$, not the customary $\gamma_0^2 I$.

The radially symmetric Cauchy distributions ($\alpha = 1$) have densities

$$f(\mathbf{x}) = \frac{\Gamma(d/2)}{2\pi^{d/2}} \frac{\gamma_0}{(\gamma_0^2 + |\mathbf{x}|^2)^{(d+1)/2}}. \tag{1.22}$$

Note that in the Gaussian case, the radially symmetric case coincides with the independent identically distributed component case. However, the radially symmetric Cauchy distribution is not the same as the independent components Cauchy distribution, which has density of the form

$$f(\mathbf{x}) = \prod_{j=1}^{d} \frac{\gamma_0}{\pi(\gamma_0^2 + x_j^2)}.$$

More generally, for any $\alpha < 2$, the independent component case does not coincide with the radially symmetric one, as the spectral measures are distinct: the spectral measure for the former has point masses whereas the latter has continuous uniform measure.

The amplitude of isotropic $\mathbf{X}$ is $R = |\mathbf{X}|$; results about the distribution of R are given in Section 2.5. Because of the radial symmetry, d-dimensional density calculations only need to be computed along a half-line, and can be expressed in terms of the distribution of R. In general, the expressions are integral formulas, but in the isotropic Gaussian and Cauchy cases, there are closed form expressions for the density and distribution function of R. In two dimensions when $\alpha = 1$, they are

$$f_R(r) = \gamma_0 r/(\gamma_0^2 + r^2)^{3/2} \quad \text{and} \quad F_R(r) = 1 - \gamma_0/\sqrt{\gamma_0^2 + r^2}.$$

Information on other dimensions and $\alpha = 2$ is given in Section 2.5. A plot of the density surface in the isotropic case is given in Figure 1.6 (top), for other values of α the radial decay is different, but the contours are all circles.

The elliptically contoured case is a linear transformation of the radially symmetric case: $\mathbf{Y} = \Sigma^{1/2}\mathbf{X}$ where Σ is a positive definite shape matrix and $\mathbf{X}$ is radially symmetric. The parameter functions are $\beta(\mathbf{u}) = 0$, $\gamma(\mathbf{u}) = (\mathbf{u}^T\Sigma\mathbf{u})^{1/2}$ and $\delta(\mathbf{u}) = 0$. There is interest in the elliptically contoured case for both theoretical and practical reasons. One interesting feature of them is that they have similar properties to the multivariate normal distributions: it is possible to express pairwise dependence of the components by a single number analogous to a covariance, and the joint dependence is completely determined by these values arranged in the symmetric positive definite matrix Σ. It is much easier to work with such distributions than with a general stable distribution. In practice, elliptical distributions come up in finance, where joint returns from some assets have an approximately elliptical joint distribution. Figure 1.6 (bottom) shows an example of the density surface for a particular shape matrix Σ.

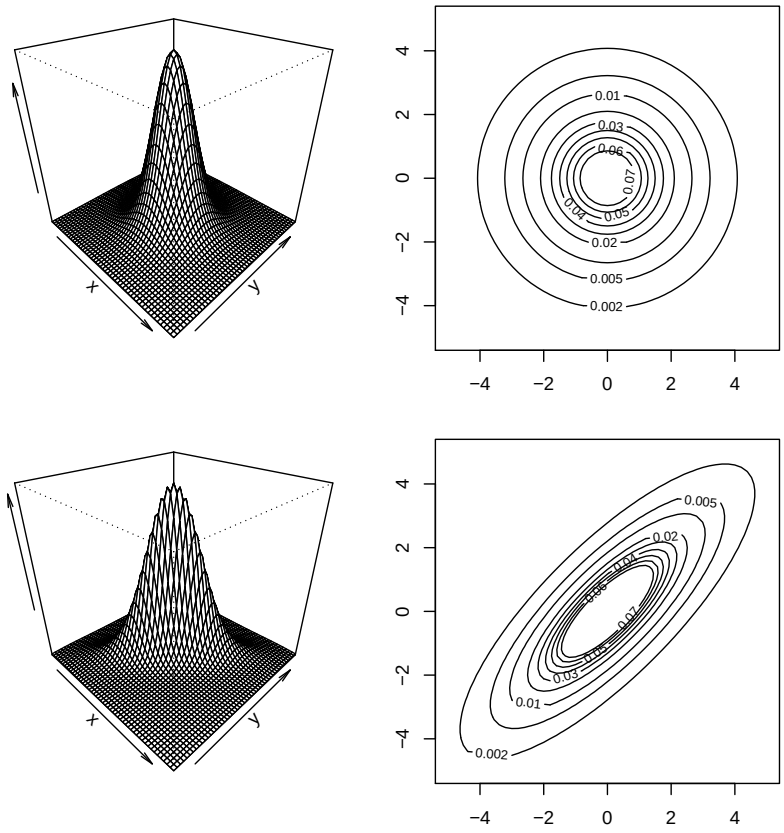

Fig. 1.6 Density surface and contours for isotropic and elliptically contoured stable laws. The top shows the isotropic case with $\alpha = 1.5$ and $\gamma_0 = 1$. The bottom shows $\alpha = 1.5$ with shape matrix $\Sigma = [1, 0.8; 0.8, 1]$.

1.4.4 Symmetric stable laws

In the multivariate case, a random vector $\mathbf{X}$ is symmetric with respect the origin if $\mathbf{X} \stackrel{d}{=} -\mathbf{X}$, i.e. for any measureable set $A \subset \mathbb{R}^d$, $P(\mathbf{X} \in A) = P(\mathbf{X} \in -A)$. This includes more than just radially symmetric, e.g. the top plots of Figures 1.3 and 1.4, both examples in Figure 1.6, and both examples in Figure 1.8. A random vector is symmetric around the point $\boldsymbol{\delta}$ if $\mathbf{X} - \boldsymbol{\delta}$ is symmetric around the origin. If the phrase symmetric random vector is used by itself, we will assume it means symmetric with respect to the origin.

In the stable case, a random vector is symmetric when the spectral measure Λ is symmetric around the origin, e.g. $\Lambda(B) = \Lambda(-B)$ for any measurable set $B \subset \mathbb{S}$. This will imply that the skewness projection function $\beta(\cdot) = 0$ for all $\mathbf{u}$. (The converse is not true in general, see UNFINISHED.) In view of Theorem 1.3 (a) and (1.12), it is sometimes useful to use $\alpha-norm$ notation, defining for $\mathbf{u} \in \mathbb{R}^d$

$$\|u_1X_1 + \cdots + u_dX_d\|_\alpha = \|\langle \mathbf{u}, \mathbf{X}\rangle\|_\alpha = \gamma(\mathbf{u}) = \left(\int_{\mathbb{S}} |\langle \mathbf{u}, \mathbf{s}\rangle|^\alpha \Lambda(d\mathbf{s})\right)^{1/\alpha}.$$

When $\alpha \geq 1$, this is a valid norm. When $0 < \alpha < 1$, it is an α quasi-norm, i.e. $\|\cdot\|_\alpha^\alpha$ satisfies has a triangle inequality even though $\|\cdot\|_\alpha$ does not. In particular, $\|X_1\|_\alpha = \gamma(1, 0, \ldots, 0)$ is the scale of X_1. Likewise $\|X_1 - X_2\|_\alpha = \gamma(1, -1, 0, \ldots, 0)$ is the scale of $X_1 - X_2$ and when the latter is small, the distributions of X_1 and X_2 are close; see Problem 1.9

In the symmetric stable case the characteristic function has a simplified form:

$$E \exp(i\langle \mathbf{u}, \mathbf{X}\rangle) = \exp\left(-\gamma^\alpha(\mathbf{u})\right) = \exp\left(-\|\langle \mathbf{u}, \mathbf{X}\rangle\|_\alpha^\alpha\right). \tag{1.23}$$

1.4.5 Sub-stable laws and linear combinations

Example 1.5 shows that if univariate P is positive α_1 stable and multivariate $\mathbf{Y}$ is strictly α_2-stable, then $\mathbf{X} = P^{1/\alpha_2}\mathbf{Y}$ is strictly $(\alpha_1\alpha_2)$-stable. For any vector $\boldsymbol{\delta}$, $P^{1/\alpha_2}\mathbf{Y}+\boldsymbol{\delta}$ is therefore stable. The precise description of $\mathbf{X}$ is given in Proposition 2.4.

If $\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n$ are α-stable random vectors (possibly dependent), then the sum $\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n$ is an α-stable random vector. For simplicity, consider the sum of two independent terms: $\mathbf{X} \stackrel{d}{=} \mathbf{Z}_1 + \mathbf{Z}_2$, where $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are independent stable random vectors. There may be advantages in modeling to consider such combinations. For example, allowing $\mathbf{Z}_1$ to have discrete spectral measure and $\mathbf{Z}_2$ to be elliptically contoured allows one to approximate more complicated dependence structures with a parsimonious model. A particular case of this is used by Rachev et al. (2003) as a model for Value-At-Risk for portfolios with d-assets. They use $\mathbf{Z}_1 = (b_1, b_2, \ldots, b_d)^T Y$, where Y is a skewed univariate stable that acts as an overall measure of the market, while $\mathbf{Z}_2$ models the dependence structure of the assets in the portfolio.

1.4.6 Complexity of the joint dependence structure

A multivariate stable distribution can have very complicated dependence structure: the fact that a spectral measure is needed to characterize the dependence means that there is no finite parameterization of multivariate stable laws. The examples discussed above give some useful subclasses with finite parameterizations. To emphasize how

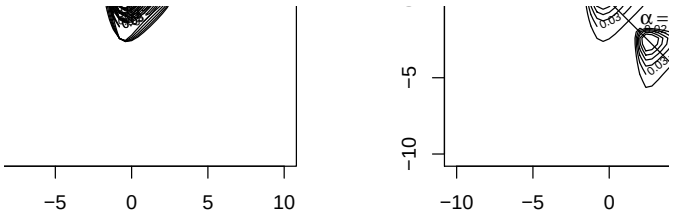

Fig. 1.7 Contours for non-symmetric stable laws with different α. Both plots show $\alpha = 0.8$, 0.9, 1.0, 1.1, 1.2 and independent components with $\beta_1 = 1$, $\beta_2 = -1$. The left plot shows the 0-parameterization, where it is seen that the contours only vary slightly as α varies. The right plot shows the 1-parameterization for α converging up to 1, the contours are centered near the diagonal line. When $\alpha = 1$, the contours are centered near the origin; when α converges down to one, the contours move toward the upper left; when α converges up to one, the contours move toward the lower right. The direction of the line is determined by the vector $\boldsymbol{\mu}_\Lambda = (1, -1)^T$ from Theorem 1.3, and the center shifts at rate $\tan \frac{\pi\alpha}{2}$. This shows that like the univariate case, the multivariate stable laws have a discontinuity at $\alpha = 1$ in the 1-parameterization.

complicated dependence can be, we give some bivariate examples. The examples below all have standardized symmetric stable (skewness $\beta = 0$, scale $\gamma = 1$, location $\delta = 0$) marginals, but distinct joint dependence. Fix $\alpha \in (0, 2)$.

!!!!!!!!!!!!!!! list hidden !!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

The first four examples above have discrete spectral measures, the fifth and sixth have absolutely continuous spectral measures, the last can have a mixture of discrete and absolutely continuous. There are many more choices for the spectral measure that will yield the same marginals: any nontrivial spectral measure will yield a distinct distribution, and that can be normalized as in the third example above to have standardized marginals. Starting with a singular measure, e.g. a Kantor measure wrapped around the circle, yields a bivariate stable distribution that is very different from any of the above.

1.5 Sums of stable random vectors

As in the univariate case, sums of α-stable terms are α-stable. This holds for all linear combinations, including multiplying by a constant matrix. The results below also show that adding dependent stable terms yield a stable law.

Recall what happens with linear combinations of multivariate Gaussian vectors. For d-dimensional $\mathbf{X} \sim N(\boldsymbol{\mu}, \Sigma)$ and $a \in \mathbb{R}$, $\mathbf{b} \in \mathbb{R}^d$,

$$a\mathbf{X} + \mathbf{b} \sim N(a\boldsymbol{\mu} + \mathbf{b}, a^2\Sigma).$$

If A is a real $p \times d$ matrix and $\mathbf{c} \in \mathbb{R}^p$,

$$A\mathbf{X} + \mathbf{c} \sim N(A\boldsymbol{\mu} + \mathbf{c}, A\Sigma A^T)$$

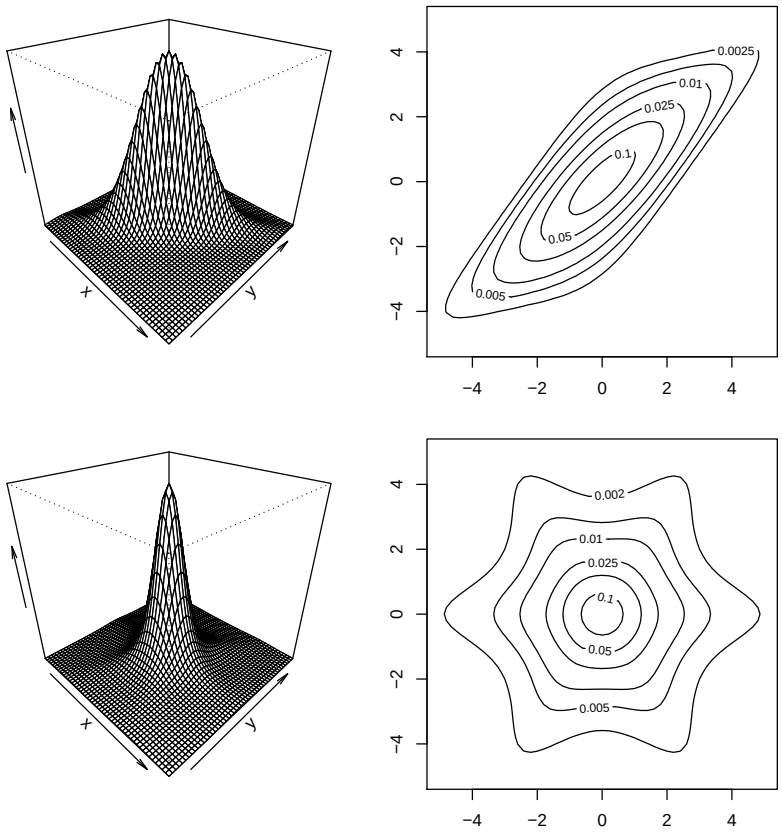

Fig. 1.8 Density surface and contours for symmetric stable laws with discrete spectral measures. The top shows the case with $\alpha = 1.7$, $\delta = (0,0)^T$ and four point masses at angles $\theta_1 = 3\pi/10$, $\theta_2 = \pi/2$, $\theta_3 = 4\pi/5$, $\theta_4 = 3\pi/2$, all of mass 1/2. The bottom shows $\alpha = 1.2, \delta = (0,0)^T$ and six point masses at angles $\theta_1 =$, $\theta_2 = \pi/3$, $\theta_3 = 2\pi/3$, $\theta_4 = \pi$, $\theta_5 = 4\pi/3$, $\theta_6 = 5\pi/3$, all of mass 1/4.

For independent $\mathbf{X}^{(1)} \sim N(\boldsymbol{\mu}_1, \Sigma_1)$ and $\mathbf{X}^{(2)} \sim N(\boldsymbol{\mu}_2, \Sigma_2)$,

$$\mathbf{X}^{(1)} + \mathbf{X}^{(2)} \sim N(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2, \Sigma_1 + \Sigma_2).$$

The following result generalizes these facts about multivariate Gaussian laws to multivariate $\alpha-$stable with $\alpha < 2$. In part (b) below, $\boldsymbol{\mu}_\Lambda$ is from Theorem 1.3 and $|a|^\alpha \Lambda((\operatorname{sign} a)\cdot)$ is the measure on $\mathbb{S}$

$$|a|^\alpha \Lambda((\operatorname{sign} a)B) = \begin{cases} |a|^\alpha \Lambda(B) & a > 0 \\ 0 & a = 0 \\ |a|^\alpha \Lambda(-B) & a < 0. \end{cases}$$

Proposition 1.2 *Let $k = 0$ or 1 and the random vectors below be d-dimensional.*
(a) If $\mathbf{X} \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); k)$, then for any $a \in \mathbb{R}$, $\mathbf{b} \in \mathbb{R}^d$,

$$a\mathbf{X} + \mathbf{b} \sim \mathbf{S}(\alpha, (\operatorname{sign} a)\beta(\cdot), |a|\gamma(\cdot), \delta(a\cdot) + \langle \cdot, \mathbf{b}\rangle; k).$$

(b) If $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; k)$, then $a\mathbf{X}+\mathbf{b} \sim \mathbf{S}(\alpha, |a|^\alpha \Lambda((\operatorname{sign} a)\cdot), a\boldsymbol{\delta} + \mathbf{b} + \mathbf{d}; k)$, where

$$\mathbf{d} = \begin{cases} -\tan\frac{\pi\alpha}{2}(a - a^{<\alpha>})\boldsymbol{\mu}_\Lambda & \alpha \neq 1, k = 0 \\ 0 & \alpha \neq 1, k = 1 \\ -\frac{2}{\pi} a \log|a| \boldsymbol{\mu}_\Lambda & \alpha = 1, k = 0, 1 \end{cases}$$

(c) If $\mathbf{X} \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); k)$ and A is a $p \times d$ real matrix and $\mathbf{c} \in \mathbb{R}^p$, then the p dimensional random vector $A\mathbf{X} + \mathbf{c}$ is stable:

$$A\mathbf{X} + \mathbf{c} \sim \mathbf{S}\left(\alpha, \beta(A^T\cdot), \gamma(A^T\cdot), \delta(A^T\cdot) + \langle \cdot, \mathbf{c}\rangle; k\right),$$

(d) If $\mathbf{X}^{(1)} \sim \mathbf{S}(\alpha, \Lambda_1, \boldsymbol{\delta}_1; k)$ and $\mathbf{X}^{(2)} \sim \mathbf{S}(\alpha, \Lambda_2, \boldsymbol{\delta}_2; k)$ are independent, then

$$\mathbf{X}^{(1)} + \mathbf{X}^{(2)} \sim \mathbf{S}(\alpha, \Lambda_1 + \Lambda_2, \boldsymbol{\delta}_1 + \boldsymbol{\delta}_2; k).$$

(e) If $\mathbf{X}^{(1)} \sim \mathbf{S}(\alpha, \beta_1(\cdot), \gamma_1(\cdot), \delta_1(\cdot); k)$ and $\mathbf{X}^{(2)} \sim \mathbf{S}(\alpha, \beta_2(\cdot), \gamma_2(\cdot), \delta_2(\cdot); k)$ are independent, then

$$\mathbf{X}^{(1)} + \mathbf{X}^{(2)} \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); k),$$

where

$$\begin{aligned} \beta(\mathbf{u}) &= \frac{\beta_1(\mathbf{u})\gamma_1^\alpha(\mathbf{u}) + \beta_2(\mathbf{u})\gamma_2^\alpha(\mathbf{u})}{\gamma_1^\alpha(\mathbf{u}) + \gamma_2^\alpha(\mathbf{u})} \\ \gamma^\alpha(\mathbf{u}) &= \gamma_1^\alpha(\mathbf{u}) + \gamma_2^\alpha(\mathbf{u}) \\ \delta(\mathbf{u}) &= \begin{cases} \delta_1(\mathbf{u}) + \delta_2(\mathbf{u}) & k = 1 \\ \delta_1(\mathbf{u}) + \delta_2(\mathbf{u}) + d_0(\mathbf{u}) & k = 0, \end{cases} \end{aligned}$$

where

$$d_0(\mathbf{u}) = \begin{cases} \tan\frac{\pi\alpha}{2}[\beta(\mathbf{u})\gamma(\mathbf{u}) - \beta_1(\mathbf{u})\gamma_1(\mathbf{u}) - \beta_2(\mathbf{u})\gamma_2(\mathbf{u})] & \alpha \neq 1, \\ \frac{2}{\pi}[\beta(\mathbf{u})\gamma(\mathbf{u})\log\gamma(\mathbf{u}) - \beta_1(\mathbf{u})\gamma_1(\mathbf{u})\log\gamma_1(\mathbf{u}) - \beta_2(\mathbf{u})\gamma_2(\mathbf{u})\log\gamma_2(\mathbf{u})] & \alpha = 1. \end{cases}$$

For sums of multiple independent stable random vectors, the above results can be used repetitively. For example, if $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(n)}$ are i.i.d. $\mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); k)$, then

$$\mathbf{X}^{(1)} + \cdots + \mathbf{X}^{(n)} \sim \mathbf{S}\left(\alpha, \beta(\cdot), n^{1/\alpha}\gamma(\cdot), n\delta(\cdot) + d_n(\cdot); k\right),$$

where

$$d_n(\mathbf{u}) = \begin{cases} \tan\frac{\pi\alpha}{2}\beta(\mathbf{u})\gamma(\mathbf{u})\left[n^{1/\alpha} - n\right] & k = 0, \alpha \neq 1 \\ 0 & otherwise. \end{cases}$$

Problem 1.10 gives a generalization of this to sums of the form $A_1\mathbf{X}^{(1)}+\cdots+A_n\mathbf{X}^{(n)}$, where $A_1, \ldots, A_n$ are matrices and $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(n)}$ are independent $\alpha-$stable random vectors, possibly of different dimensions.

This result shows that the sum of dependent jointly stable terms is stable, and to describe the sum, the dependence structure must be taken into account. In the univariate setting, this is done by treating the univariate summands $X_1, \ldots, X_d$ as a jointly stable random vector. Suppose $\mathbf{X} = (X_1, X_2, \ldots, X_d)^T \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); k)$. Then Proposition 1.2(c) with $A = \mathbf{1} = (1, 1, \ldots, 1)$ shows

$$X_1 + X_2 + \cdots + X_d = A\mathbf{X} \sim \mathbf{S}(\alpha, \beta(\mathbf{1}), \gamma(\mathbf{1}), \delta(\mathbf{1}); k). \tag{1.24}$$

Consider the $d = 2$ case when $\mathbf{X} = (X_1, X_2)^T$ symmetric $\alpha-$stable. Then

$$(\text{scale of } X_1 + X_2)^\alpha = \gamma^\alpha(1, 1) = \gamma^\alpha(1, 0)+\gamma^\alpha(0, 1)+\left[\gamma^\alpha(1, 1) - \gamma^\alpha(1, 0) - \gamma^\alpha(0, 1)\right].$$

Or in the norm notation of Section 1.4.4,

$$\|X_1 + X_2\|_\alpha^\alpha = \|X_1\|_\alpha^\alpha + \|X_2\|_\alpha^\alpha + \left[\|X_1 + X_2\|_\alpha^\alpha - \|X_1\|_\alpha^\alpha - \|X_2\|_\alpha^\alpha\right].$$

In both versions, the term in brackets determines the difference between $\|X_1 + X_2\|_\alpha^\alpha$ and $\|X_1\|_\alpha^\alpha + \|X_2\|_\alpha^\alpha$. The term in brackets has a name: it is minus the co-difference of X_1 and $-X_2$, written $-\tau_{X_1,-X_2}$. When $\alpha = 2$ it is minus twice the covariance of X_1 and X_2. When $0 < \alpha < 2$, the addition rule for scales is

$$\|X_1 + X_2\|_\alpha^\alpha = \|X_1\|_\alpha^\alpha + \|X_2\|_\alpha^\alpha - \tau_{X_1,-X_2}. \tag{1.25}$$

Section 3.2.2 shows that the co-difference is zero if X_1 and X_2 are independent, but that there are also dependent cases with a co-difference of zero.

A similar statement holds for the sum of dependent stable random vectors that are jointly stable. View the collection as an (nd)-dimensional dependent stable distribution: $\mathbf{X} = (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(n)}) \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); k)$. Let $A = (I, I, \ldots, I)$, the $d \times (nd)$ matrix composed of blocks of $d \times d$ identity matrices I, then Proposition 1.2(c) shows

$$\mathbf{X}^{(1)} + \mathbf{X}^{(2)} + \cdots + \mathbf{X}^{(n)} = A\mathbf{X} \sim \mathbf{S}(\alpha, \beta_1(\cdot), \gamma_1(\cdot), \delta_1(\cdot); k), \tag{1.26}$$

is d dimensional α-stable with parameter projection functions $\beta_1(\mathbf{u}) = \beta(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u})$, $\gamma_1(\mathbf{u}) = \gamma(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u})$, and $\delta(\mathbf{u}) = \delta_1(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u})$.

1.6 Simulation

It is not known how to simulate an arbitrary multivariate stable vector, but several of the classes discussed above can be simulated exactly.

Modarres and Nolan (1994) gave a method of simulating when the spectral measure Λ is discrete. When $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; k)$ when Λ is concentrated on the points $\mathbf{s}_1, \ldots, \mathbf{s}_n$ with $\Lambda(\{\mathbf{s}_j\}) = \lambda_j$, then (1.6) shows

$$\mathbf{X} \stackrel{d}{=} \sum_{j=1}^{m} \lambda_j^{1/\alpha} Z_j \mathbf{s}_j + \boldsymbol{\delta},$$

where Z_j are univariate $\mathbf{S}(\alpha, 1; k)$ random variables. If $k = 1$ and $\boldsymbol{\delta} = \mathbf{0}$, then $\mathbf{X}$ is strictly stable.

Example 1.3 gives a formula for simulating sub-Gaussian/elliptically contoured stable random vectors: let $\mathbf{G} = (G_1, ..., G_d) \sim N(0, \Sigma)$, $0 < \alpha < 2$, and $R \sim \mathbf{S}(\alpha/2, 1; 1)$, then

$$\mathbf{X} := R^{1/2}\mathbf{G} + \boldsymbol{\delta}$$

is an elliptically contoured α-stable random vector. If $\boldsymbol{\delta} = \mathbf{0}$, then this $\mathbf{X}$ is strictly stable.

Likewise, if $\mathbf{Y}$ is any strictly α_1-stable random vector and univariate R is positive α_2-stable, then $\mathbf{X} := R^{1/\alpha_1}\mathbf{Y}$ is strictly $(\alpha_1\alpha_2)$-stable. In particular, the two cases discussed above have strictly stable subclasses that can be used for $\mathbf{Y}$.

TEXT INCOMPLETE/HIDDEN

Finally, linear combinations of terms that can be simulated using the above methods are multivariate stable, as in Proposition 1.2.

It is possible to approximately simulate multivariate stable laws by summing vectors in the domain of attraction of stable laws, see Davydov and Nagaev (2002a) and Davydov and Nagaev (2002b). Another approximate approach is to use a Poisson part for large and moderate jumps and a Brownian approximation for small jumps, see Asmussen and Rosiński (2001).

1.7 Multivariate generalized central limit theorem

When $\mathbf{X} = (X_1, \ldots, X_d)^T$ has components with finite variance, then $\boldsymbol{\mu} = (EX_1, \ldots, EX_d)^T$ is finite and for all pairs i, j, $\sigma_{ij} = \mathrm{Cov}(X_i, X_j)$ is finite. The classical Central Limit Theorem states that in this case,

$$n^{-1/2}(\mathbf{X}^{(1)} + \mathbf{X}^{(2)} + \cdots + \mathbf{X}^{(n)} - n\boldsymbol{\mu}) \stackrel{d}{\longrightarrow} \mathbf{Z} \sim N(0, \Sigma),$$

where $\Sigma = [\sigma_{ij}]_{i,j=1,\ldots,d}$. A proof of this can be found in Chapter 5 of Ferguson (1996). It is also possible to weaken the assumption that the terms are identical, see the Lindeberg-Feller Theorem in this last reference. Finally, in the univariate case,

Theorem I.3.13 shows that a finite variance is not necessary to get a normal limit; a similar result is possible in the multivariate case, see Section 8.1 of Meerschaert and Scheffler (2001).

When the components of $\mathbf{X}$ have heavy enough tails, the classical Central Limit Theorem with a normal limit does not hold. As in the univariate case, the only possible non-trivial limits of appropriately normalized sums are the stable laws. Section 8.2 of Meerschaert and Scheffler (2001) proves the following two results.

Theorem 1.6 ***Multivariate Generalized Central Limit Theorem*** *A nondegenerate random vector* $\mathbf{Z}$ *is* $\alpha-$*stable for some* $0 < \alpha \leq 2$ *if and only if there is an independent, identically distributed sequence of random vectors* $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots$ *and constants* $a_n > 0$, $\mathbf{b}_n \in \mathbb{R}^d$ *with*

$$a_n(\mathbf{X}^{(1)} + \cdots + \mathbf{X}^{(n)}) - \mathbf{b}_n \xrightarrow{d} \mathbf{Z}.$$

As in the univariate case, we must have $a_n \approx c/n^{1/\alpha}$ for large n. In the univariate case, Theorem I.3.14 gives necessary and sufficient condtions for a r.v. to be in the domain of attraction of a stable law. In that theorem, it sufficed to look at the behavior of both $P(X_1 > r$ and $P(-X_1 > r)$ as $r \to \infty$. In the multivariate case, we need to verify not only tail behavior along the axes, but also along all directions. Here is the correct multivariate statement using the definition of a cone from Section 1.3.

Theorem 1.7 ***Multivariate Generalized Central Limit Theorem with limit identified*** *A random vector* $\mathbf{X}$ *belongs to the domain of attraction of a stable distribution* $\mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; 1)$ *if and only if both the following hold:*
(a)$P(|\mathbf{X}| > r)$ *is regularly varying of index* $-\alpha$ *as* $r \to \infty$.
(b) For every Borel set $B \subset \mathbb{S}$ *with* $\Lambda(\partial B) = 0$,

$$\frac{P(|\mathbf{X}| > r, \mathbf{X} \in \text{Cone}(B))}{P(|\mathbf{X}| > r)} \to \frac{\Lambda(B)}{\Lambda(\mathbb{S})} \quad as \quad r \to \infty.$$

1.8 Problems

Problem 1.1 Let Z_1, Z_2, Z_3 be i.i.d. $\mathbf{S}(\alpha, 1; 1)$, $0 < \alpha < 2$.
(a) Set $\mathbf{t}_1 = (1, 0)^T$, $\mathbf{t}_2 = (-1, 1)^T$ and $\mathbf{t}_3 = (0, -1)^T$. Show that $\mathbf{X} = Z_1\mathbf{t}_1 + Z_2\mathbf{t}_2 + Z_3\mathbf{t}_3 = (Z_1 - Z_2, Z_2 - Z_3)^T$ has both components symmetric, but $\mathbf{X}$ is a non-symmetric stable random vector.
(b) Set $\mathbf{t}_1 = (1, 0)^T$, $\mathbf{t}_2 = (-1/2, \sqrt{3}/2)^T$ and $\mathbf{t}_3 = (-1/2, -\sqrt{3}/2)^T$. Show that $\mathbf{X} = Z_1\mathbf{t}_1 + Z_2\mathbf{t}_2 + Z_3\mathbf{t}_3$ is a non-symmetric stable random vector with every one dimensional projection symmetric if $\alpha = 1$, but most projections non-symmetric if $\alpha \neq 1$.

Problem 1.2 Show that the vector $\mathbf{X}$ in Example 1.4 is not jointly stable. More generally, for a d-dmensional random vector, all components being univariate stable

with the same α is a necessary, but not sufficient, condition for the vector to be jointly stable.

Problem 1.3 Let Z_1, Z_2 be i.i.d. univariate $\mathbf{S}(\alpha, 1; 1)$.
(a) Define $\mathbf{X} = (Z_1, 0)^T$. This is a degenerate two dimensional stable vector, with probability concentrated on the x-axis. Show that $\gamma(\mathbf{u}) = 0$ when $\mathbf{u}$ is on the y-axis. Further $\beta(\mathbf{u}) = +1$ for any $\mathbf{u}$ in the right half-plane and $\beta(\mathbf{u}) = -1$ for any $\mathbf{u}$ in the left half-plane, so there is an unremovable discontinuity in $\beta(\mathbf{u})$ along the y-axis.
(b) Define $\mathbf{X} = (Z_1, Z_2, 0)^T$, a degenerate three dimensional stable vector, with probability concentrated on the xy plane. Show that $\gamma(\mathbf{u}) = 0$ on the z-axis and the limit of $\beta(\mathbf{u})$ as $\mathbf{u}$ approaches this line is undefined. In particular, if $x > 0$, $y > 0$, $z \neq 0$, then $\beta(x, y, z) = +1$, $\beta(-x, -y, z) = -1$, $-1 < \beta(-x, y, z) < +1$, and $-1 < \beta(x, -y, z) < +1$.

Problem 1.4 Spectral measures are not unique when $\alpha = 2$. Let $X \sim N(0, \Sigma)$ be a two dimensional normal random vector with

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Set $\mathbf{s}_1 = (1, 0)^T$, $\mathbf{s}_2 = (0, 1)^T$, $\mathbf{s}_3 = (1/\sqrt{2}, 1/\sqrt{2})^T$, $\mathbf{s}_4 = (1/\sqrt{1+\rho^2}, \rho/\sqrt{1+\rho^2})^T$, $\mathbf{s}_5 = (1/\sqrt{1+\rho^2}, -\rho/\sqrt{1+\rho^2})^T$. Show that

$$\Lambda_1(\cdot) = \frac{1-\rho}{4}\left[\mathbf{1}_{\{\mathbf{s}_1 \in \cdot\}} + \mathbf{1}_{\{-\mathbf{s}_1 \in \cdot\}} + \mathbf{1}_{\{\mathbf{s}_2 \in \cdot\}} + \mathbf{1}_{\{-\mathbf{s}_2 \in \cdot\}}\right] + \frac{\rho}{2}\left[\mathbf{1}_{\{\mathbf{s}_3 \in \cdot\}} + \mathbf{1}_{\{-\mathbf{s}_3 \in \cdot\}}\right]$$

$$\Lambda_2(\cdot) = \frac{\sqrt{1-\rho^2}}{4}\left[\mathbf{1}_{\{\mathbf{s}_2 \in \cdot\}} + \mathbf{1}_{\{-\mathbf{s}_2 \in \cdot\}}\right] + \frac{1+\rho^2}{4}\left[\mathbf{1}_{\{\mathbf{s}_4 \in +\}}\mathbf{1}_{\{\mathbf{s}_5 \in \cdot\}}\right]$$

are distinct discrete measures that give the same characteristic function in (1.8).

Problem 1.5 Show the following properties of $\boldsymbol{\mu}_\Lambda$ from Theorem 1.3.
(a) $\boldsymbol{\mu}_\Lambda \in [-\Lambda(\mathbb{S}), \Lambda(\mathbb{S})]^d$.
(b) Find spectral measures that make $\boldsymbol{\mu}_\Lambda$ achieve each point in $[-\Lambda(\mathbb{S}), \Lambda(\mathbb{S})]^d$.
(c) If Λ is symmetric, then $\boldsymbol{\mu}_\Lambda = \mathbf{0}$.
(d) If Λ is supported in some orthant, then $\boldsymbol{\mu}_\Lambda$ is in that orthant.
(e) When $\alpha = 1$, $\boldsymbol{\mu}_\Lambda = (\beta(\mathbf{e}_1)\gamma(\mathbf{e}_1), \ldots, \beta(\mathbf{e}_d)\gamma(\mathbf{e}_d))^T$, the vector $\mathbf{b}$ in Proposition 1.1.

Problem 1.6 A multivariate stable density can have very different tail behavior in different directions.
(a) In the two dimensional symmetric independent case, $f(x_1, x_2) = f_1(x_1)f_1(x_2)$, where f_1 is a univariate stable density. Along either axes, $f(0, x) = f(x, 0) = f_1(0)f_1(x)$, and for x large, $f(0, x) = f(x, 0) \sim cx^{-(1+\alpha)}$. On the other hand, for any angle $\theta \in (0, \pi/2)$, the density in direction $(\cos\theta, \sin\theta)$ is $f(x\cos\theta, x\sin\theta) = f_1(x\cos\theta)f_1(x\sin\theta) \sim c(x\cos\theta)^{-(1+\alpha)}c(x\sin\theta)^{-(1+\alpha)} = c^2(\cos\theta\sin\theta)^{-(1+\alpha)}x^{-2(1+\alpha)}$. Hence the radial decay is much faster in the direction $(\cos\theta, \sin\theta)$ than along either axes.

(b) Radially symmetric case, $f(\mathbf{x}) = g(|x|)$ where $g(\cdot)$ is specified in Section 2.5. There it is shown that $g(|x|) \sim c|x|^{-(d+\alpha)}$ for large $|x|$, is the same in all directions.

Problem 1.7 Let $\mathbf{Y} = A\mathbf{X}$ where A is a $d \times d$ matrix and $\mathbf{X}$ has independent components $X_j \sim \mathbf{S}\left(\alpha, \beta_j, \gamma_j, \delta_j; k\right)$, $j = 1, \ldots, d$. Find the spectral measure of $\mathbf{Y}$ and the parameter functions $\beta(\cdot), \gamma(\cdot)$ and $\delta(\cdot)$. Show $\mathbf{Y}$ has a density if and only if A is invertible, in which case the density can be expressed in terms of univariate stable densities as in (1.21).

Problem 1.8 For a multivariate stable density $f(\mathbf{x})$ and a direction $\mathbf{u} \in \mathbb{S}_d$, $x \in \mathbb{R}$, define $g_{\mathbf{u}}(x) = f(x\mathbf{u})$. The graph of $(x, g_{\mathbf{u}}(x))$ is a section of the multivariate graph $(\mathbf{x}, f(\mathbf{x}))$ along the direction $\mathbf{u}$. In the $\alpha = 2$ Gaussian case, every section $g_{\mathbf{u}}(\cdot)$ is an unnormalized univariate Gaussian density. Use Problem 1.6 to show that in contrast, the section of a stable law with $0 < \alpha < 2$ is generally not a univariate stable law.

Problem 1.9 Let $(X_1, X_2)^T$ be bivariate symmetric $\alpha-$stable, $Z \sim \mathbf{S}(\alpha, 0, 1, 0; 0)$, and $\epsilon > 0$.
(a) Show $P(|X_1 - X_2| < \epsilon) = P(|Z| < \epsilon / \|X_1 - X_2\|_\alpha)$.
(b) Show $P(|X_1 - X_2| < \epsilon) \sim 1 - 2c_\alpha \|X_1 - X_2\|_\alpha^\alpha / \epsilon^\alpha$, as $\|X_1 - X_2\|_\alpha \to 0$.(hint hidden)

Problem 1.10 Let $\mathbf{X}^{(j)} \sim \mathbf{S}\left(\alpha, \beta_j(\cdot), \gamma_j(\cdot), \delta_j(\cdot); k\right)$ be a d_j dimensional stable random vector and A_j is a $p \times d_j$ real matrix for $j = 1, \ldots, n$, and all $\mathbf{X}^{(j)}$ are independent, then $A_1\mathbf{X}^{(1)} + A_2\mathbf{X}^{(2)} + \cdots + A_n\mathbf{X}^{(n)}$ is a p dimensional α-stable random vector with parameter functions

$$\begin{aligned}
\gamma^\alpha(\mathbf{u}) &= \sum_{j=1}^n \gamma_j^\alpha(A_j^T\mathbf{u}) \\
\beta(\mathbf{u}) &= \left(\sum_{j=1}^n \beta_j(A_j^T\mathbf{u})\gamma_j^\alpha(A_j^T\mathbf{u})\right) / \gamma^\alpha(\mathbf{u}) \\
\delta(\mathbf{u}) &= \sum_{j=1}^n \delta_j(A_j^T\mathbf{u}) + d_*(\mathbf{u}),
\end{aligned}$$

where

$$d_*(\mathbf{u}) = \begin{cases} 0 & k = 1 \\ \tan\frac{\pi\alpha}{2}\left[\beta(\mathbf{u})\gamma(\mathbf{u}) - \sum_{j=1}^n \beta_j(A_j^T\mathbf{u})\gamma_j(A_j^T\mathbf{u})\right] & k = 0, \alpha \neq 1, \\ \frac{2}{\pi}\left[\beta(\mathbf{u})\gamma(\mathbf{u})\log\gamma(\mathbf{u}) - \sum_{j=1}^n \beta_j(A_j^T\mathbf{u})\gamma_j(A_j^T\mathbf{u})\log\gamma_j(A_j^T\mathbf{u})\right] & k = 0, \alpha = 1. \end{cases}$$

Problem 1.11 Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be elliptically contoured $\alpha-$stable, $0 < \alpha \leq 2$, with respective shape matrices R_1 and R_2.
(a) If $\alpha = 2$, then $\mathbf{X}_1 + \mathbf{X}_2$ is elliptically contoured with shape matrix $R_1 + R_2$.
(b) If $\alpha < 2$, then $\mathbf{X}_1 + \mathbf{X}_2$ is generally not elliptically contoured.

(c) If $R_2 = cR_1$ for some $c > 0$ and any α, then $\mathbf{X}_1 + \mathbf{X}_2$ is elliptically contoured with shape matrix $(1 + c^{\alpha/2})^{2/\alpha} R_1$.